Asymptotic stability of harmonic maps under the Schrödinger flow *

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Abstract

For Schrödinger maps from $\mathbb{R}^2 \times \mathbb{R}^+$ to the 2-sphere $\mathbb{S}^2$, it is not known if finite energy solutions can form singularities ("blowup") in finite time. We consider equivariant solutions with energy near the energy of the two-parameter family of equivariant harmonic maps. We prove that if the topological degree of the map is at least four, blowup does *not* occur, and global solutions converge (in a dispersive sense – i.e. scatter) to a fixed harmonic map as time tends to infinity. The proof uses, among other things, a time-dependent splitting of the solution, the "generalized Hasimoto transform", and Strichartz (dispersive) estimates for a certain two space-dimensional linear Schrödinger equation whose potential has critical power spatial singularity and decay. Along the way, we establish an energy-space local well-posedness result for which the existence time is determined by the length-scale of a nearby harmonic map.

Contents

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1 Introduction and main results

The *Schrödinger flow* for maps from $\mathbb{R}^n$ to $\mathbb{S}^2$ (also known as the *Schrödinger map*, and, in ferromagnetism, as the *Heisenberg model* or *Landau-Lifshitz equation*) is given by the equation

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times \Delta \mathbf{u}, \qquad \mathbf{u}(x,0) = \mathbf{u}_0(x). \tag{1.1}$$

Here $\mathbf{u} = \mathbf{u}(x,t)$ is the unknown map from $\mathbb{R}^n \times \mathbb{R}^+$ to the 2-sphere

$$\mathbb{S}^2 := \{\mathbf{u} \in \mathbb{R}^3 \ \mid \ |\mathbf{u}| = 1\} \subset \mathbb{R}^3,$$

Δ denotes the Laplacian in $\mathbb{R}^n$, and $\times$ denotes the cross product of vectors in $\mathbb{R}^3$. A somewhat more geometric way of writing Equation (1.1) is

$$\frac{\partial \mathbf{u}}{\partial t} = JP\Delta \mathbf{u} \tag{1.2}$$

where $P = P^{\mathbf{u}}$ denotes the orthogonal projection from $\mathbb{R}^3$ onto the tangent plane

$$T_{\mathbf{u}}\mathbb{S}^2 := \{\boldsymbol{\xi} \in \mathbb{R}^3 \mid \boldsymbol{\xi} \cdot \mathbf{u} = 0\}$$

to $\mathbb{S}^2$ at $\mathbf{u}$ (so that $P\Delta\mathbf{u} = \Delta\mathbf{u} + |\nabla \mathbf{u}|^2\mathbf{u}$), and

$$J = J^{\mathbf{u}} := \mathbf{u}\times$$

is a rotation through $\pi/2$ on the tangent plane $T_{\mathbf{u}}\mathbb{S}^2$.

On one hand, Equation (1.1) is a borderline case of the *Landau-Lifshitz-Gilbert* equations which model dynamics in isotropic ferromagnets (including dissipation):

$$\frac{\partial \mathbf{u}}{\partial t} = aP\Delta\mathbf{u} + bJP\Delta\mathbf{u}, \qquad a \geq 0 \tag{1.3}$$

(see, eg., [15]). The Schrödinger flow corresponds to the case $a = 0$. The case $b = 0$ is the well-studied harmonic map heat flow, for which some finite-energy solutions do blow up in finite time ([4]).

On the other hand, Equation (1.1) is a particular case of the Schrödinger flow for maps from a Riemannian manifold into a Kähler manifold (see, eg., [8, 25, 10, 7]). We will consider only the case of maps : $\mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{S}^2$ in this paper.

We refer the reader to our previous paper [11] for more detailed background on (1.1) (and further references), limiting the discussion here to a list of a few basic facts we need in order to state our results.

- **Energy conservation.** Equation (1.1) formally conserves the *energy*

$$\mathcal{E}(\mathbf{u}) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla \mathbf{u}|^2 \, dx = \frac{1}{2}\int_{\mathbb{R}^n} \sum_{j=1}^{n}\sum_{k=1}^{3} \left|\frac{\partial u_k}{\partial x_j}\right|^2 dx. \tag{1.4}$$

  The space dimension $n = 2$ is critical in the sense that $\mathcal{E}(\mathbf{u})$ is invariant under scaling. In general,

$$\mathcal{E}(\mathbf{u}(\cdot)) = s^{2-n}\mathcal{E}(\mathbf{u}(\cdot/s)) \tag{1.5}$$

  for $s > 0$.

- **Equivariant maps.** Fix $m \in \mathbb{Z}$ a non-zero integer. By an *m-equivariant map* $\mathbf{u} : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$, we mean a map of the form

$$\mathbf{u}(r,\theta) = e^{m\theta R}\,\mathbf{v}(r) \tag{1.6}$$

  where (r,θ) are polar coordinates on $\mathbb{R}^2$, $\mathbf{v} : [0,\infty) \to \mathbb{S}^2$, and R is the matrix generating rotations around the u_3-axis:

$$R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e^{\alpha R} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.7}$$

  Radial maps arise as the case $m = 0$. The class of m-equivariant maps is formally preserved by the Schrödinger flow.

- **Topological lower bound on energy.** If $\mathbf{u}$ is m-equivariant, we have $|\nabla \mathbf{u}|^2 = |\partial \mathbf{u}/\partial r|^2 + r^{-2}|\partial \mathbf{u}/\partial\theta|^2 = |\partial \mathbf{v}/\partial r|^2 + (m^2/r^2)|R\mathbf{v}|^2$ and so

$$\mathcal{E}(\mathbf{u}) = \pi\int_0^\infty \left(\left|\frac{\partial \mathbf{v}}{\partial r}\right|^2 + \frac{m^2}{r^2}(v_1^2 + v_2^2) \right) r dr. \tag{1.8}$$

  If $\mathcal{E}(\mathbf{u}) < \infty$, then $\mathbf{v}(r)$ is continuous, and the limits $\lim_{r\to 0}\mathbf{v}(r)$ and $\lim_{r\to\infty}\mathbf{v}(r)$ exist (see [11]), and so we must have $\mathbf{v}(0), \mathbf{v}(\infty) = \pm\widehat{\mathbf{k}}$, where $\widehat{\mathbf{k}} = (0,0,1)^T$. Without loss of generality we fix $\mathbf{v}(0) = -\widehat{\mathbf{k}}$. The two cases $\mathbf{v}(\infty) = \pm\widehat{\mathbf{k}}$ then correspond to different topological classes of maps. We denote by Σ_m the class of m-equivariant maps with $\mathbf{v}(\infty) = \widehat{\mathbf{k}}$:

$$\Sigma_m = \left\{ \mathbf{u} : \mathbb{R}^2 \to \mathbb{S}^2 \;\middle|\; \mathbf{u} = e^{m\theta R}\mathbf{v}(r),\;\; \mathcal{E}(\mathbf{u}) < \infty,\;\; \mathbf{v}(0) = -\widehat{\mathbf{k}},\;\; \mathbf{v}(\infty) = \widehat{\mathbf{k}} \right\}. \tag{1.9}$$

For $\mathbf{u} \in \Sigma_m$, the energy $\mathcal{E}(\mathbf{u})$ can be rewritten:

$$\mathcal{E}(\mathbf{u}) = \pi \int_0^\infty \left(\left| \frac{\partial \mathbf{v}}{\partial r} \right|^2 + \frac{m^2}{r^2} |J^{\mathbf{v}} R \mathbf{v}|^2 \right) r dr = \pi \int_0^\infty \left| \frac{\partial \mathbf{v}}{\partial r} - \frac{|m|}{r} J^{\mathbf{v}} R \mathbf{v} \right|^2 r dr + \mathcal{E}_{\min} \tag{1.10}$$

(recall $J^{\mathbf{v}} := \mathbf{v} \times$) with

$$\mathcal{E}_{\min} = 2\pi \int_0^\infty \mathbf{v}_r \cdot \frac{|m|}{r} J^{\mathbf{v}} R \mathbf{v} \, r dr = 2\pi |m| \int_0^\infty (v_3)_r dr = 4\pi |m|. \tag{1.11}$$

Thus for $\mathbf{u} \in \Sigma_m$, there is a non-trivial lower bound for the energy:

$$\mathbf{u} \in \Sigma_m \quad \Longrightarrow \quad \mathcal{E}(\mathbf{u}) \geq 4\pi |m|. \tag{1.12}$$

(In general one has $\mathcal{E}(\mathbf{u}) \geq 4\pi |deg|$ where deg is the *degree* of the map $\mathbf{u}$, considered as a map from $\mathbb{S}^2$ to itself (defined, for example, by integrating the pullback by $\mathbf{u}$ of the volume form on $\mathbb{S}^2$).)

- **Harmonic maps.** For a map $\mathbf{u} \in \Sigma_m$, the topological lower bound (1.12) is saturated if and only if

$$\frac{\partial \mathbf{v}}{\partial r} = \frac{|m|}{r} J^{\mathbf{v}} R \mathbf{v}, \tag{1.13}$$

and the minimal energy is attained (i.e. (1.13) is satisfied) precisely at the two-parameter family of harmonic maps

$$\mathcal{O}_m := \left\{ e^{(m\theta + \alpha)R} \mathbf{h}(r/s) \;\middle|\; s > 0, \; \alpha \in \mathbb{R} \right\} \tag{1.14}$$

where

$$\mathbf{h}(r) = \begin{pmatrix} h_1(r) \\ 0 \\ h_3(r) \end{pmatrix}, \quad h_1(r) = \frac{2}{r^{|m|} + r^{-|m|}}, \quad h_3(r) = \frac{r^{|m|} - r^{-|m|}}{r^{|m|} + r^{-|m|}}. \tag{1.15}$$

The rotation parameter α is determined only up to shifts of 2π (i.e. really $\alpha \in \mathbb{S}^1$). The fact that $\mathbf{h}(r)$ satisfies (1.13) means

$$(h_1)_r = -\frac{m}{r} h_1 h_3, \quad (h_3)_r = \frac{m}{r} h_1^2. \tag{1.16}$$

Note that $\mathcal{O}_m$ is just the orbit of the harmonic map $e^{m\theta R}\mathbf{h}(r)$ under the symmetries of the energy $\mathcal{E}$ which preserve equivariance: scaling and rotation. Explicitly, the maps in $\mathcal{O}_m$ are of the form

$$u(r, \theta) = \begin{pmatrix} \cos(m\theta + \alpha) h_1(r/s) \\ \sin(m\theta + \alpha) h_1(r/s) \\ h_3(r/s) \end{pmatrix}. \tag{1.17}$$

Of course, these harmonic maps are each static solutions of the Schrödinger flow (1.1). In fact, it is not hard to show they are the only m-equivariant static solutions (though this fact plays no role in our analysis).

- **The "orbital stability" of $\mathcal{O}_m$.** We recall the main result of [11]:

Theorem 1.1 *[11] There exist $\delta > 0$ and $C_1, C_2 > 0$ such that if $\mathbf{u} \in C([0,T); \dot{H}^2 \cap \Sigma_m)$ is a solution of the Schrödinger flow* (1.1) *conserving energy, and satisfying*

$$\delta_1^2 := \mathcal{E}(\mathbf{u}_0) - 4\pi|m| < \delta^2,$$

then there exist $s(t) \in \mathcal{C}([0,T);(0,\infty))$ and $\alpha(t) \in \mathcal{C}([0,T);\mathbb{R})$ so that

$$\left\|\mathbf{u}(x,t) - e^{(m\theta+\alpha(t))R}\mathbf{h}(r/s(t))\right\|_{\dot{H}_1(\mathbb{R}^2)} \leq C_1\delta_1, \quad \forall t \in [0,T). \tag{1.18}$$

Moreover, $s(t) > C_2/\|\mathbf{u}(t)\|_{\dot{H}_2(\mathbb{R}^2)}$. Furthermore, if $T < \infty$ is the maximal time of existence for $\mathbf{u}$ in $\dot{H}^2$ (i.e. if $\lim_{t\to T^-}\|\mathbf{u}(t)\|_{\dot{H}^2(\mathbb{R}^2)} = \infty$), then

$$\liminf_{t\to T^-} s(t) = 0. \tag{1.19}$$

This theorem can be viewed, on one hand, as an *orbital stability* result for the family $\mathcal{O}_m$ of harmonic maps (at least up to the possible blowup time), and on the other hand as a characterization of blowup for energy near $\mathcal{E}_{\min}$: solutions blowup if and only if the "length-scale" $s(t)$ goes to zero. Here $s(t)$ (and the rotation angle $\alpha(t)$) are determined simply by finding, at each time t, the harmonic map which is $\dot{H}^1$-closest to $\mathbf{u}(t)$. More precisely, a continuous map

$$\begin{aligned} \{\ \mathbf{u} \in \Sigma_m \mid \mathcal{E}(\mathbf{u}) < 4\pi|m| + \delta^2\ \} &\to \mathbb{R}^+ \times (\mathbb{R} \mod 2\pi) \\ \mathbf{u} &\mapsto (\ s(\mathbf{u}),\ \alpha(\mathbf{u})\ ) \end{aligned} \tag{1.20}$$

is constructed in [11], which, for m-equivariant maps with energy close to $4\pi|m|$, identifies the unique $\dot{H}^1$-closest harmonic map:

$$\|\mathbf{u} - e^{[m\theta+\alpha(\mathbf{u})]R}h(r/s(\mathbf{u}))\|_{\dot{H}^1} = \min_{s\in\mathbb{R}^+,\alpha\in\mathbb{R}} \|\mathbf{u} - e^{[m\theta+\alpha]R}h(r/s)\|_{\dot{H}^1}. \tag{1.21}$$

Then we set $s(t) := s(\mathbf{u}(t))$.

In this paper, we continue our study of the Schrödinger flow for equivariant maps with energy close to the harmonic map energy. We begin with an energy-space local well-posedness theorem for such maps. It is worth remarking that despite a great deal of recent work on the local well-posedness problem in two space dimensions ([23, 9, 17, 1, 16]; see also [18, 12, 14] for the "modified Schrödinger map" case), there is no general result for energy space initial data. For our special class of data, however, we do have such a result. Before stating it, let us first make precise the sense in which our energy-space solution solves the Schrödinger map problem:

Definition 1.2 (Weak solutions) *Let $Z := \{\ \mathbf{u} : \mathbb{R}^n \to \mathbb{S}^2,\ D\mathbf{u} \in L^2\ \}$ be the energy space. We say $\mathbf{u}(x,t)$ is a weak solution of the Schrödinger flow* (1.1) *on the time interval $I = [0,T]$, with initial data $\mathbf{u}_0 \in Z$, if*

1. $\mathbf{u} \in L^\infty(I; Z) \cap C_{weak}([0,T]; Z)$

2. $\mathbf{u}(0) = \mathbf{u}_0$

3. $\iint_{\mathbb{R}^n \times I} \{\mathbf{u} \cdot \boldsymbol{\phi}_t - \mathbf{u} \times \partial_j \mathbf{u} \cdot \partial_j \boldsymbol{\phi}\} dxdt = 0$ *for all* $\boldsymbol{\phi} \in C_c^1(I \times \mathbb{R}^n; \mathbb{R}^3)$.

Remark 1.3 It is not strictly necessary to require that $D\mathbf{u}$ be weakly continuous in t (in property 1 above). The weak form of the equation (property 3) implies $\mathbf{u}_t \in L^\infty([0,T]; H^{-1})$, and so, after redefinition on a set of time measure zero, $\mathbf{u} \in Lip([0,T]; H^{-1})$ and $D\mathbf{u} \in Lip([0,T]; H^{-2})$. Since we also have $D\mathbf{u} \in L^\infty([0,T]; L^2)$, we can prove $D\mathbf{u} \in C_{weak}([0,T]; L^2)$.

We have

Theorem 1.4 (Local wellposedness) *Let* $|m| \geq 1$. *There exist* $\delta > 0$ *and* $\sigma > 0$ *such that the following hold. Suppose* $\mathbf{u}_0 \in \Sigma_m$ *and* $\mathcal{E}(\mathbf{u}_0) = 4\pi m + \delta_0^2$, $\delta_0 \in (0, \delta]$. *Let* $s_0 := s(\mathbf{u}_0)$, *as defined in* (1.20)-(1.21). *Then there is a unique weak solution* $\mathbf{u}(t)$ *of* (1.1)

$$\mathbf{u}(t) \in C(I; \Sigma_m), \quad I = [0, \sigma s_0^2].$$

Moreover, $\mathcal{E}(\mathbf{u}(t)) = \mathcal{E}(\mathbf{u}_0)$ *for* $t \in I$. *If, furthermore,* $\mathbf{u}_0 \in \dot{H}^2$, *then* $\mathbf{u}(t) \in C(I; \Sigma_m \cap \dot{H}^2)$. *Suppose* $\mathbf{u}_0^n \to \mathbf{u}_0$ *in* Σ_m *and let* $\mathbf{u}^n$ *denote the corresponding solutions of* (1.1), *then* $\mathbf{u}^n \to \mathbf{u}$ *in* $C(I, \Sigma_m)$.

It is worth emphasizing that the existence time furnished by this theorem depends not on the energy $\|\mathbf{u}_0\|_{\dot{H}^1}^2$ of the initial data (reflecting the energy-space critical nature of the equation in dimension $n = 2$), but rather on $s(\mathbf{u}_0)$, the length scale of the $\dot{H}^1$-nearest harmonic map.

There are at least two ways to define blow-up for these solutions. Suppose $\mathbf{u}(t) \in C([0,T), \Sigma_m \cap \dot{H}^k)$, $0 < T < \infty$ with $k = 1$ or 2. If $k = 1$, we say $\mathbf{u}(t)$ blows up at $t = T$ if $\lim_{t \to T^-} \mathbf{u}(t)$ does not exist in $\dot{H}^1$. If $k = 2$, we say $\mathbf{u}(t)$ blows up at $t = T$ if $\limsup_{t \to T-} \|\mathbf{u}(t)\|_{\dot{H}^2} = \infty$.

For $\mathbf{u}_0 \in \Sigma_m \cap \dot{H}^k$, $k = 1, 2$, denote by $T_{\max}^k$ the maximal time such that there is a unique solution $\mathbf{u}(t) \in C([0, T_{\max}^k); \Sigma_m \cap \dot{H}^k)$.

Corollary 1.5 *Under the same assumptions as in Theorem 1.4, suppose the solution* $\mathbf{u}(t) \in C([0,T), \Sigma_m \cap \dot{H}^k)$, $k = 1$ *or* 2, *and* $T < \infty$.

(i) *(Blowup alternative)* $\mathbf{u}(t)$ *blows up at time* T *(i.e.* $T = T_{max}^1$*) iff* $\liminf_{t \to T-} s(\mathbf{u}(t)) = 0$. *In this case,* $s(\mathbf{u}(t)) \leq C\sqrt{T-t}$, *and if* $k = 2$, $T = T_{max}^1 = T_{max}^2$ *with* $\|\mathbf{u}(t)\|_{\dot{H}^2} \geq C(T-t)^{-1/2}$.

(ii) *(Lower bound for* $T_{\max} := T_{max}^1$*) We have* $T_{\max} \geq \sigma[s(\mathbf{u}_0)]^2$ *(here* σ *is the constant from Theorem 1.4).*

Corollary 1.5 (i) improves Theorem 1.1 by giving explicit bounds.

We also have $\dot{H}^1$ local wellposedness for the small energy equivariant case considered in [5]. Since the energy is conserved, local wellposedness implies global wellposedness.

Theorem 1.6 (Small energy local wellposedness) *Let $|m| \geq 1$. There exist $\delta > 0$ and $\sigma > 0$ such that the following hold. Suppose $\mathbf{u}_0 = e^{m\theta R} v_0(r)$ and $\mathcal{E}(\mathbf{u}_0) \leq \delta^2$, then there is a unique weak solution $\mathbf{u}(t,r,\theta) = e^{m\theta R}\mathbf{v}(t,r)$ of (1.1) so that $\mathbf{u}(t) \in C([0,\sigma]; \dot{H}^1)$. Moreover, $\mathcal{E}(\mathbf{u}(t)) = \mathcal{E}(\mathbf{u}_0)$ for $t \in [0,\sigma]$. Suppose $\mathbf{u}_0^n$ are equivariant, $\mathbf{u}_0^n \to \mathbf{u}_0$ in $\dot{H}^1$ and let $\mathbf{u}^n$ denote the corresponding solutions of (1.1), then $\mathbf{u}^n \to \mathbf{u}$ in $C([0,\sigma], \dot{H}^1)$.*

Note that this result does *not* cover the radial case ($m = 0$).

The question of whether singularities can form in the Schrödinger flow is open. So far, it has only been shown that they cannot form for *small energy* radial or equivariant solutions ([5]). Our Theorem 1.1 above leaves open the question of whether finite-time blowup can occur for maps in Σ_m with energies near $\mathcal{E}_{\min} = 4\pi|m|$. The main result of this paper shows that when $|m| \geq 4$, it does *not*. Moreover, we show that these solutions converge (in a dispersive sense) to specific harmonic maps as $t \to \infty$. Here is the main result:

Theorem 1.7 (Main result) *Let $|m| \geq 4$. Let (r,p) satisfy $2 < r \leq \infty$, $2 \leq p < \infty$, with $1/r + 1/p = 1/2$. There exist positive constants δ, C, and C_p, such that if $\mathbf{u}_0 \in \Sigma_m$ satisfies*

$$\delta_1^2 := \mathcal{E}(\mathbf{u}_0) - 4\pi|m| < \delta^2,$$

then for the corresponding solution $\mathbf{u}(t)$ of the Schrödinger flow (guaranteed by Theorem 1.4),

1. *there is no finite-time blowup: $T_{\max} = \infty$*
2. *there exist $s(t) \in \mathcal{C}([0,\infty); (0,\infty))$ and $\alpha(t) \in \mathcal{C}([0,\infty); \mathbb{R})$ such that*

$$\left\| \nabla[\mathbf{u}(x,t) - e^{(m\theta + \alpha(t))R}\mathbf{h}(r/s(t))] \right\|_{(L_t^\infty L_x^2 \cap L_t^r L_x^p)(\mathbb{R}^2 \times [0,\infty))} \leq C_p \delta_1 \tag{1.22}$$

3. *furthermore,*

$$\left| \frac{s(t)}{s(\mathbf{u}_0)} - 1 \right| + |\alpha(t) - \alpha(\mathbf{u}_0)| \leq C\delta_1^2 \qquad \textit{for all } \ t > 0$$

and there exist $s_+ > 0$ and α_+ with

$$s(t) \to s_+, \quad \alpha(t) \to \alpha_+, \quad \textit{as} \quad t \to \infty. \tag{1.23}$$

Remark 1.8 1. The $L_t^\infty L_x^2$ (energy space) estimate in (1.22) already follows from Theorem 1.1. The other space-time estimates in (1.22) further imply asymptotic *convergence* to the family of harmonic maps (at least, in a time-averaged sense – the best we can expect without further assumptions on the initial data). The convergence results (1.22) and (1.23) are precisely what we mean when we say the harmonic maps are *asymptotically stable* under the Schrödinger flow for $|m| \geq 4$.

2. Note that for $|m| = 1, 2, 3$, the fate of solutions with energy near $\mathcal{E}_{\min}$ is still an open question. Our restriction $|m| > 3$ is connected with the slow spatial decay of the harmonic map component $h_1(r) \sim (const)r^{-|m|}$ as $r \to \infty$. For a somewhat technical reason, we need $r^2 h_1(r) \in L^2(rdr)$ (see Lemma 2.3), which requires $|m| > 3$. For seemingly more fundamental reasons, we need $rh_1(r) \in L^2(rdr)$ (see (2.17)), which holds if $|m| > 2$.

3. The recent work [22] on the analogous *wave map* problem, imposes the same $|m| \geq 4$ restriction, but proves that *blow-up* is possible in this class, suggesting that singularity formation is a more delicate question for Schrödinger maps than for wave maps.

We end the introduction with a few words about our approach. One key observation, already used in [11], is that the tangent vector field

$$\mathbf{W} := \frac{\partial \mathbf{v}}{\partial r} - \frac{|m|}{r} J^{\mathbf{v}} R\mathbf{v}$$

"measures the deviation of the map $\mathbf{u}$ from harmonicity" (this is indicated by (1.13), for example). Furthermore, when expressed in an appropriate orthonormal frame, the coordinates of $\mathbf{W}$ satisfy a nonlinear Schrödinger-type equation which is suitable for obtaining estimates – this is the *generalized Hasimoto transform* introduced in [5] to study the small energy problem.

In the present work, this nonlinear Schrödinger-type PDE is coupled to a two-dimensional dynamical system describing the dynamics of the scaling and rotation parameters $s(t)$ and $\alpha(t)$, a careful choice of which must be made at each time in order to allow estimation. This is all done in Section 2.

The key to proving convergence of the solution to a harmonic map is then to obtain dispersive estimates – in this case Strichartz-type estimates – for the linear part of our nonlinear Schrödinger equation. The potential appearing in the corresponding Schrödinger operator turns out to have $const/|x|^2$ behaviour both at the origin, and as $|x| \to \infty$, which is a "borderline" case not treatable by purely perturbative methods. Fortunately, a recent series of papers by Burq, Planchon, Stalker, and Tahvildar-Zadeh (see [2, 3]) addresses the problem of obtaining dispersive estimates when the potential has just this "critical" decay rate, provided the potential satisfies a "repulsivity" condition (which in particular rule out bound states). Though their relevant results are for dimension $n \geq 3$, we are able to adapt their approach to prove the estimates we need in our two-dimensional setting. This is done in Section 3.

Finally, in Section 4, we prove Theorem 1.7 by applying the linear estimates of Section 3 to the coupled nonlinear system of Section 2.

Since the proof of Theorems 1.4 and 1.6, and Corollary 1.5 are independent of the rest of the paper, they are postponed to Section 5. Some lemmas are proved in Section 6.

Remark 1.9 1. From here on, we will assume $m > 0$. For $m < 0$, simply make the change of variable $(x_1, x_2, x_3) \to (x_1, -x_2, x_3)$.

2. **Notation:** throughout the paper, the letter C is used to denote a generic constant, the value of which may change from line to line. Vectors in $\mathbb{R}^3$ appear in boldface, while their components appear in regular type: for example, $\mathbf{u} = (u_1, u_2, u_3)$.

2 The dynamics near the harmonic maps

2.1 Splitting the solution

Let $\mathbf{u}(x,t) = e^{m\theta R}\mathbf{v}(r,t) \in \Sigma_m$ be a solution of the Schrödinger map equation (1.1). We will write our solution as a harmonic map with time-varying parameters, plus a perturbation:

$$\mathbf{v}(r,t) = e^{\alpha(t)R}\left[\mathbf{h}(\rho) + \boldsymbol{\xi}(\rho,t)\right], \qquad \rho := \frac{r}{s(t)} \tag{2.1}$$

In Section 2.3 we take up the central question of precisely how to do this splitting (i.e. the choice of $s(t)$ and $\alpha(t)$).

It is convenient and natural to single out the component of the perturbation $\boldsymbol{\xi}$ which is tangent to $\mathbb{S}^2$ at $\mathbf{h}$:

$$\boldsymbol{\xi}(\rho,t) = \boldsymbol{\eta}(\rho,t) + \gamma(\rho,t)\mathbf{h}(\rho), \qquad \boldsymbol{\eta}(\rho,t) \in T_{\mathbf{h}(\rho)}\mathbb{S}^2,$$

so that $\boldsymbol{\eta}\cdot\mathbf{h} \equiv 0$. Thus the original map $\mathbf{u}$ is written

$$\mathbf{u}(x,t) = e^{[m\theta+\alpha(t)]R}[(1+\gamma(\rho,t))\mathbf{h}(\rho) + \boldsymbol{\eta}(\rho,t)]; \qquad \rho = \frac{r}{s(t)}, \qquad \boldsymbol{\eta}(\rho,t) \in T_{\mathbf{h}(\rho)}\mathbb{S}^2.$$

The pointwise constraint $|\mathbf{v}| \equiv 1$ forces

$$1 \equiv |\mathbf{h}+\boldsymbol{\xi}|^2 = |(1+\gamma)\mathbf{h}+\boldsymbol{\eta}|^2 = (1+\gamma)^2 + |\boldsymbol{\eta}|^2,$$

so $\gamma(\rho,t) \le 0$ and $|\boldsymbol{\eta}(\rho,t)| \le 1$. If $|\boldsymbol{\xi}| \le 1$, then

$$\gamma(\rho,t) = +(1-|\boldsymbol{\eta}(\rho,t)|^2)^{1/2} - 1 \in [-1,0], \tag{2.2}$$

A convenient orthonormal basis of $T_{\mathbf{h}(\rho)}\mathbb{S}^2$ is given by

$$\hat{\boldsymbol{\jmath}} := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad J^{\mathbf{h}(\rho)}\hat{\boldsymbol{\jmath}} = \begin{pmatrix} -h_3(\rho) \\ 0 \\ h_1(\rho) \end{pmatrix},$$

and we will express tangent vectors like $\boldsymbol{\eta} \in T_{\mathbf{h}}\mathbb{S}^2$ in this basis via the invertible linear map

$$\mathbf{V}^\rho : \mathbb{C} \to T_{\mathbf{h}(\rho)}\mathbb{S}^2$$
$$z = z_1 + iz_2 \mapsto z_1\hat{\boldsymbol{\jmath}} + z_2 J^{\mathbf{h}(\rho)}\hat{\boldsymbol{\jmath}}.$$

So we write

$$\boldsymbol{\eta}(\rho,t) = \mathbf{V}^\rho(z(\rho,t)),$$

and in this way, the complex function $z(\rho, t)$, together with a choice of the parameters $s(t)$ and $\alpha(t)$, gives a full description of the original solution $\mathbf{u}(x,t)$, provided $|\boldsymbol{\xi}| \leq 1$.

From (2.2), we find

$$|z| = |\boldsymbol{\eta}| \leq 1/2 \quad \Longrightarrow \quad |\gamma| \lesssim |z|^2, \quad |\gamma_\rho| \lesssim |z||z_\rho|. \tag{2.3}$$

These estimates, together with results in [11], show that if s and α are chosen appropriately, then for $\mathcal{E}(\mathbf{u}) - 4\pi m$ small,

$$\|z\|_X^2 \lesssim \mathcal{E}(\mathbf{u}) - 4\pi m \lesssim \|z\|_X^2$$

where $X := \{z : [0,\infty) \to \mathbb{C} \mid z_\rho \in L^2(\rho d\rho), \ \frac{z}{\rho} \in L^2(\rho d\rho)\}$, with

$$\|z\|_X^2 := \int_0^\infty \left\{ |z_\rho(\rho)|^2 + \frac{|z(\rho)|^2}{\rho^2} \right\} \rho d\rho. \tag{2.4}$$

The space X is therefore the natural space for z, corresponding to the energy space for the original map $\mathbf{u}$. The facts

$$z \in X \implies z \text{ continuous in } (0,\infty), \ z(0+) = z(\infty-) = 0, \text{ and } \|z\|_{L^\infty} \lesssim \|z\|_X, \tag{2.5}$$

follow easily from the change of variable $\rho^m = e^y$ and Sobolev imbedding on $\mathbb{R}$ (see [11]).

2.2 Equation for the perturbation

The next step is to derive an equation for $z(\rho,t)$. In terms of $\mathbf{v}(r,t)$, the Schrödinger map equation can be written as

$$\mathbf{v}_t = \mathbf{v} \times \left(\mathbf{v}_{rr} + \frac{1}{r}\mathbf{v}_r + \frac{m^2}{r^2} R^2 \mathbf{v} \right). \tag{2.6}$$

Using (2.1), we find

$$e^{-\alpha R}\mathbf{v}_t = [\dot{\alpha} R - s^{-1}\dot{s}\rho\partial_\rho](\mathbf{h} + \boldsymbol{\xi}) + \boldsymbol{\xi}_t, \tag{2.7}$$

$$s^2 e^{-\alpha R}(\mathbf{v} \times M_r \mathbf{v}) = (\mathbf{h} + \boldsymbol{\xi}) \times (M_\rho \mathbf{h} + M_\rho \boldsymbol{\xi}), \tag{2.8}$$

where

$$M_\rho := \partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{m^2}{\rho^2} R^2$$

(and the right-hand sides are evaluated at $(\rho = r/s(t), t)$).

Consider first (2.8). Since $\Delta\mathbf{H} + |\nabla \mathbf{H}|^2 \mathbf{H} = 0$ for $\mathbf{H} = e^{m\theta R}\mathbf{h}$, we have

$$M\mathbf{h} = -2\frac{m^2}{\rho^2} h_1^2 \mathbf{h}, \tag{2.9}$$

where $M = M_\rho$. Thus,

$$\begin{aligned}\text{RHS of (2.8)} &= \mathbf{h} \times M\boldsymbol{\xi} + \boldsymbol{\xi} \times (-2\frac{m^2}{\rho^2} h_1^2 \mathbf{h}) + \boldsymbol{\xi} \times M\boldsymbol{\xi} \\ &= \mathbf{h} \times (M + 2\frac{m^2}{\rho^2} h_1^2)\boldsymbol{\xi} + \boldsymbol{\xi} \times M\boldsymbol{\xi}.\end{aligned}$$

Keeping in mind (2.3), we write

$$\text{RHS of (2.8)} = \mathbf{h} \times (M + 2\frac{m^2}{\rho^2} h_1^2)(\mathbf{V}^\rho(z)) + \mathbf{F_1}$$

where $\mathbf{F_1} = \mathbf{h} \times (M + 2\frac{m^2}{\rho^2} h_1^2)\gamma\mathbf{h} + \boldsymbol{\xi} \times M\boldsymbol{\xi}$ is the nonlinear part. By (2.9), we have $M\gamma\mathbf{h} = 2\gamma_\rho \mathbf{h}_\rho + (\cdots)\mathbf{h} = 2\gamma_\rho \frac{m}{\rho}\widehat{\mathbf{k}} + (\cdots)\mathbf{h}$, and hence

$$\mathbf{F_1} = -2\gamma_\rho \frac{m}{\rho} h_1 \widehat{\boldsymbol{\jmath}} + \boldsymbol{\xi} \times M\boldsymbol{\xi}. \tag{2.10}$$

Using $R^2\widehat{\boldsymbol{\jmath}} = -\widehat{\boldsymbol{\jmath}}$, $R^2 J^{\mathbf{h}}\widehat{\boldsymbol{\jmath}} = h_1 h_3 \mathbf{h} - h_3^2 J^{\mathbf{h}}\widehat{\boldsymbol{\jmath}}$, $(J^{\mathbf{h}}\widehat{\boldsymbol{\jmath}})_\rho = -\frac{m}{\rho} h_1 \mathbf{h}$, and $(J^{\mathbf{h}}\widehat{\boldsymbol{\jmath}})_{\rho\rho} = -\frac{m^2}{\rho^2} h_1^2 J^{\mathbf{h}}\widehat{\boldsymbol{\jmath}} - (\frac{m}{\rho} h_1)_\rho \mathbf{h}$ (all easy computations), we find that the linear part can be rewritten as

$$\mathbf{h} \times (M + 2\frac{m^2}{\rho^2} h_1^2)(\mathbf{V}^\rho(z))) = -\mathbf{h} \times [\mathbf{V}^\rho(Nz)] = \mathbf{V}^\rho(-iNz)$$

where N denotes the differential operator $N := -\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \frac{m^2}{\rho^2}(1 - 2h_1^2)$.

Because $\boldsymbol{\xi}_t = \mathbf{V}^\rho(z_t) + \gamma_t \mathbf{h}$, (2.7)–(2.8) give

$$s^2[\mathbf{V}^\rho(z_t) + \gamma_t \mathbf{h}] + [s^2\dot{\alpha}R - s\dot{s}\rho\partial_\rho](\mathbf{h} + \boldsymbol{\xi}) = \mathbf{V}^\rho(-iNz) + \mathbf{F_1},$$

or

$$\mathbf{V}^\rho(s^2 z_t + iNz) = \mathbf{F}, \tag{2.11}$$

where

$$\mathbf{F} := \mathbf{F_1} + [-s^2\dot{\alpha}R + s\dot{s}\rho\partial_\rho](\mathbf{h} + \boldsymbol{\xi}) - s^2\gamma_t \mathbf{h}.$$

Because the l.h.s. of (2.11) is $\in T_{\mathbf{h}}\mathbb{S}^2$, the r.h.s is also, and hence $\mathbf{F} \cdot \mathbf{h} \equiv 0$. We can re-write (2.11) on the complex side by applying $(V^\rho)^{-1}$:

$$is^2 \frac{\partial z}{\partial t} = Nz + i(\mathbf{V}^\rho)^{-1}\mathbf{F}, \qquad N = -\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \frac{m^2}{\rho^2}(1 - 2h_1^2). \tag{2.12}$$

This is the equation we sought for $z(\rho, t)$.

In order to see the form of the "nonlinear" terms $(\mathbf{V}^\rho)^{-1}(\mathbf{F})$ more clearly, we compute

$$(\mathbf{V}^\rho)^{-1}(R\mathbf{h}(\rho)) = h_1(\rho), \quad (\mathbf{V}^\rho)^{-1}(\rho\partial_\rho \mathbf{h}(\rho)) = imh_1,$$
$$(\mathbf{V}^\rho)^{-1}(P^{\mathbf{h}(\rho)} R\mathbf{V}^\rho(z)) = izh_3, \quad (\mathbf{V}^\rho)^{-1}(P^{\mathbf{h}(\rho)} \rho\partial_\rho \mathbf{V}^\rho(z)) = \rho z_\rho,$$

where $P^{\mathbf{h}(\rho)}$ denotes the orthogonal vector projection onto $T_{\mathbf{h}(\rho)}\mathbb{S}^2$. Thus, using $\mathbf{h} + \boldsymbol{\xi} = (1 + \gamma)\mathbf{h} + \mathbf{V}^\rho(z)$,

$$(\mathbf{V}^\rho)^{-1}(\mathbf{F}) = [-s^2\dot{\alpha} + ims\dot{s}](1 + \gamma)h_1 - s^2\dot{\alpha}izh_3 + s\dot{s}\rho z_\rho + (\mathbf{V}^\rho)^{-1}(P^{\mathbf{h}(\rho)}\mathbf{F_1}). \tag{2.13}$$

2.3 Orthogonality condition and parameter equations

We have not yet specified $s(t)$ and $\alpha(t)$. The main result of [11] says that if the energy is close to $\mathcal{E}_{\min}$, that is $\delta_1^2 := \mathcal{E}(\mathbf{u}) - \mathcal{E}_{min} \ll 1$, then there exist continuous $s(t) > 0$ and $\alpha(t) \in \mathbb{R}$ such that $\|e^{m\theta R}\boldsymbol{\xi}\|_{\dot{H}^1} \lesssim \delta_1$ as long as $s(t)$ stays away from 0. The choice of the parameters was simple and natural: at each time t, $s(t)$ and $\alpha(t)$ were chosen so as to minimize $\|e^{m\theta R}\boldsymbol{\xi}\|_{\dot{H}^1}$. In this paper, we are forced into a different choice of $s(t)$ and $\alpha(t)$, as we shall now explain.

Supposing for a moment that $s(t) \equiv 1$, the linearized equation for $z(\rho, t)$ can be read from (2.12):

$$i\partial_t z = Nz. \tag{2.14}$$

The factorization

$$N = L_0^* L_0, \qquad L_0 := \partial_\rho + \frac{m}{\rho} h_3 = h_1 \partial_\rho \frac{1}{h_1} \tag{2.15}$$

(where the adjoint L_0^* is taken in the $L^2(\rho d\rho)$ inner product) shows that $ker N = span\{h_1\}$. In particular, (2.14) admits the constant (in time) solution $z(\rho, t) \equiv h_1(\rho)$. Since we would like $z(\rho, t)$ to have some decay in time, we must choose $s(t)$ and $\alpha(t)$ in such a way as to avoid such constant solutions. Since N is self-adjoint in L^2, the natural choice is to work in the subspace of functions z satisfying

$$(z,\ h_1)_{L^2} = \int_0^\infty z(\rho) h_1(\rho) \rho d\rho \equiv 0, \tag{2.16}$$

which is invariant under the linear flow (2.14).

Recall, however, that the "energy space" for z is the space X (defined in (2.4)). Certainly the linear flow (2.14) does *not* preserve the subspace $\{f \in X, \langle f, h_1 \rangle_X = 0\}$ (since N is not self-adjoint in X). In fact, neither z nor h_1 lies in L^2 in general. The best we can do is

$$|(z, h_1)_{L^2}| = \left| \left(\frac{z}{\rho}, \rho h_1 \right)_{L^2} \right| \le \|z\|_X \|\rho h_1\|_{L^2}.$$

So to make sense of (2.16), we require

$$\rho h_1(\rho) = \frac{2\rho}{\rho^m + \rho^{-m}} \in L^2(\rho d\rho), \tag{2.17}$$

which only holds if $m \ge 3$. This is one of the reasons we cannot handle the small $|m|$ cases in Theorem 1.7. The further restriction $m > 3$ is needed in Proposition 2.2 to come.

In order to ensure condition (2.16) holds for all times t, it suffices to impose it initially, and then ensure the time derivative of the inner-product vanishes for all t. Differentiating (2.16) with respect to t, and using Equations (2.12), (2.13), and (2.16), yields a system of ODEs for $s(t)$ and $\alpha(t)$:

$$[s^2\dot{\alpha} - ims\dot{s}](h_1,\ (1+\gamma)h_1)_{L^2} = (h_1,\ (\mathbf{V}^\rho)^{-1}(P^{\mathbf{h}(\rho)}\mathbf{F_1}) - s^2\dot{\alpha} i h_3 z + s\dot{s}\rho z_\rho)_{L^2}. \tag{2.18}$$

The orthogonality condition (2.16) is precisely the one that ensures the terms linear in z disappear from 2.18, and hence the key property that $\dot{s}$ and $\dot{\alpha}$ be at least *quadratic* in z. More precisely, the system (2.18) leads to the following estimate:

Lemma 2.1 *If* $\|z\|_X \ll 1$, *then*

$$|s\dot{s}| + |s^2\dot{\alpha}| \lesssim \left\|\frac{z}{\rho^2}\right\|_{L^2}^2 + \left\|\frac{z_\rho}{\rho}\right\|_{L^2}^2.$$

Proof. Using

$$\begin{aligned} |(h_1, h_3 z)| &\lesssim \|\rho h_1\|_{L^2}\|z/\rho\|_{L^2} \lesssim \|z\|_X \ll 1 \\ |(h_1, \rho z_\rho)| &\lesssim \|\rho h_1\|_{L^2}\|z_\rho\|_{L^2} \lesssim \|z\|_X \ll 1 \\ |(h_1, \gamma h_1)| &\lesssim \|\rho^2 h_1^2\|_{L^\infty}\|z/\rho\|_{L^2}^2 \lesssim \|z\|_X^2 \ll 1, \end{aligned}$$

in (2.18), we arrive at

$$|s\dot{s}| + |s^2\dot{\alpha}| \lesssim |(h_1, (\mathbf{V}^\rho)^{-1}(P^{\mathbf{h}(\rho)}\mathbf{F_1}))|. \tag{2.19}$$

To finish the proof of the lemma, we will need to find $(\mathbf{V}^\rho)^{-1}(P^{\mathbf{h}}\mathbf{F_1})$ explicitly. Using the calculation of Lemma 6.1 in Appendix B, we have

$$\begin{aligned} (h_1, (\mathbf{V}^\rho)^{-1}P^{\mathbf{h}}\mathbf{F_1})_{L^2} = \int_0^\infty \bigg(& i(h_1)_\rho(-\gamma z_\rho + z\gamma_\rho) + \frac{m}{\rho}h_1^2(-2\gamma_\rho - iz_2(z_1)_\rho + iz_1(z_2)_\rho) \\ & + \frac{m}{\rho}(h_1^2)_\rho(\gamma^2 - iz_2 z) + i\frac{m^2}{\rho^2}(2h_1^2 - 1)h_1\gamma z\bigg)\rho d\rho. \end{aligned}$$

Now using the inequality (2.5), together with $(h_1)_\rho = -(m/\rho)h_1h_3$, and the fact that $\rho^2 h_1(\rho)$ is bounded for $m \geq 2$, the estimate

$$|(h_1, (\mathbf{V}^\rho)^{-1}(P^{\mathbf{h}}\mathbf{F_1})_{L^2}| \lesssim \left\|\frac{z}{\rho^2}\right\|_{L^2}^2 + \left\|\frac{z_\rho}{\rho}\right\|_{L^2}^2$$

follows. Together with (2.19), this completes the proof of Lemma 2.1. □

2.4 A nonlinear Schrödinger equation suited to estimates

We need to prove that $z(\rho, t)$ has some decay in time, but the nonlinear Schrödinger-type equation (2.12) is not suitable for obtaining such estimates, for at least two reasons. Firstly, as remarked previously, the linearized equation has constant solutions, and so the orthogonality condition (2.16) has to be explicitly used in order to get any decay whatsoever. Secondly, and maybe more seriously, some of the nonlinear terms contain derivatives (even two derivatives) of z, leading to a loss of regularity. Fortunately, there is a neat way around these problems: the *generalized Hasimoto transform* of [5] yields an equation without these difficulties, as we now explain.

Let $\mathbf{u} = e^{m\theta R}\mathbf{v}(r) \in \Sigma_m$. From (1.10), it is clear that the tangent vector

$$\mathbf{W}(r) := \mathbf{v}_r(r) - \frac{m}{r} J^{\mathbf{v}} R\mathbf{v}(r) \quad \in T_{\mathbf{v}(r)}\mathbb{S}^2$$

plays a distinguished role. In particular, $\mathbf{u}$ is a harmonic map if and only if $\mathbf{W} \equiv 0$. Indeed, the Schrödinger map equation (1.1), written in terms of $\mathbf{v}(r,t)$, can be factored as

$$\frac{\partial \mathbf{v}}{\partial t} = J^{\mathbf{v}}[D_r^{\mathbf{v}} + \frac{1}{r} - \frac{m}{r} v_3]\mathbf{W} \tag{2.20}$$

where

$$D_r^{\mathbf{v}} := P^{\mathbf{v}(\mathbf{r})}\partial_r$$

denotes the *covariant derivative* (with respect to r, along $\mathbf{v}$). The idea is to write an equation for $\mathbf{W}$ in an appropriate intrinsic way.

Following [5], let $\mathbf{e}(r) \in T_{\mathbf{v}(r)}\mathbb{S}^2$ be a unit-length tangent field satisfying the "gauge condition"

$$D_r^{\mathbf{v}}\mathbf{e} \equiv 0. \tag{2.21}$$

Expressing $\mathbf{W}$ in the orthonormal frame $\{\mathbf{e},\ J^{\mathbf{v}}\mathbf{e}\}$,

$$\mathbf{W} = q_1\mathbf{e} + q_2 J^{\mathbf{v}}\mathbf{e},$$

and using (2.20), and (2.21), it is not difficult to arrive at the following equation for the complex function $q(r,t) := q_1(r,t) + iq_2(r,t)$:

$$\begin{aligned} iq_t &= -(\partial_r + \frac{m}{r}v_3)(\partial_r + \frac{1}{r} - \frac{m}{r}v_3)q + Sq \\ &= (-\Delta_r + \frac{1}{r^2}((1-mv_3)^2 + mr(v_3)_r))q + Sq \end{aligned} \tag{2.22}$$

where the function $S(r,t)$ arises as $D_t^{\mathbf{v}}\mathbf{e} = SJ^{\mathbf{v}}\mathbf{e}$. From the curvature relation

$$[D_r, D_t]\mathbf{e} = -Re\left[\left(\partial_r + \frac{1}{r} - \frac{m}{r}v_3\right) q\ \overline{\left(q + \frac{m}{r}\nu\right)}\right] J^{\mathbf{v}}\mathbf{e},$$

where $P^{\mathbf{v}(\mathbf{r})}\widehat{\mathbf{k}} = \widehat{\mathbf{k}} - v_3\mathbf{v} = \nu_1\mathbf{e} + \nu_2 J^{\mathbf{v}}\mathbf{e}$, we find

$$S = Re\int_r^\infty \left(\partial_\tau + \frac{1}{\tau} - \frac{m}{\tau}v_3(\tau,t)\right) q(\tau,t)\ \overline{\left(q(\tau,t) + \frac{m}{\tau}\nu(\tau,t)\right)} d\tau. \tag{2.23}$$

Thus the term in (2.22) involving S is non-local and nonlinear. We can simplify the expression for S by integrating by parts in the term involving $\partial_\tau q$, and using the relation $\nu_r = -v_3(q + (m/r)\nu)$, to arrive at

$$S(r,t) = -\frac{1}{2}Q(r,t) + \int_r^\infty \frac{1}{\tau}Q(\tau,t)d\tau, \qquad Q := |q|^2 + \frac{2m}{r}Re(\bar{\nu}q). \tag{2.24}$$

Thus Equation (2.22) resembles a cubic nonlinear Schrödinger equation, keeping in mind (a) there are non-local nonlinear terms, and (b) it is not self-contained: the unknown map $\mathbf{v}(r,t)$ itself appears in several places (including through ν). Furthermore, since

$$\delta_1^2 = \mathcal{E}(\mathbf{u}) - 4\pi m = \frac{1}{2}\|\mathbf{W}\|_{L^2}^2 = \pi\|q\|_{L^2(rdr)}^2,$$

we are dealing with a *small L^2-data problem* for Equation (2.22) (even though the map $\mathbf{u}$ is not a small-energy map). This is what allows us the estimates we need.

Because of the fact (b) mentioned above, and in order to close the estimate of Lemma 2.1, we need to be able to control z (and hence $\mathbf{v}$) in terms of q. This is only possible if we have a supplementary condition such as (2.16) (since $q=0$ just means $\mathbf{v}(r) = e^{\alpha R}h(r/s)$ for some s, α). Parts of the proof of the following estimates are a simple adaptation of the corresponding argument in [11], where the orthogonality condition was somewhat different.

Proposition 2.2 *If $m \geq 3$ and (2.16) holds, and if $\|z\|_X \ll 1$, then for $2 \leq p < \infty$,*

1. $\|z_\rho\|_{L^p} + \left\|\frac{z}{\rho}\right\|_{L^p} \lesssim s^{1-2/p}\|q\|_{L^p}$
2. *if* $m > 3$, $\left\|\frac{z_\rho}{\rho}\right\|_{L^2} + \left\|\frac{z}{\rho^2}\right\|_{L^2} \lesssim s\left\|\frac{q}{r}\right\|_{L^2}$.

Proof. The first observation is that, modulo nonlinear terms, $q(r)$ is equivalent to $(1/s)(L_0 z)(r/s)$, where $L_0 = \partial_\rho + \frac{m}{\rho}h_3(\rho)$. Precisely,

$$\begin{aligned} s\mathbf{W}(s\rho) =& \mathbf{V}^\rho(L_0 z) + \frac{m}{\rho}z_1(h_1 z_2 + h_3\gamma)\hat{\jmath} \\ & + \frac{m}{\rho}(-h_1 z_1^2 + [h_3 z_2 - h_1(1+\gamma)]\gamma)J^{\mathbf{h}}\hat{\jmath} + (\gamma_\rho + \frac{m}{\rho}[h_1 z_2\gamma - h_3|z|^2])\mathbf{h}. \end{aligned}$$

Using (2.5), it follows easily that for $2 \leq p \leq \infty$,

$$\begin{aligned} \|L_0 z\|_{L^p} &\lesssim s^{1-2/p}\|q\|_{L^p} + (\|z\|_X + \|z\|_X^3)\||z_\rho| + |z|/\rho\|_{L^p} \\ \|\frac{1}{\rho}L_0 z\|_{L^2} &\lesssim s\|\frac{1}{r}q\|_{L^2} + (\|z\|_X + \|z\|_X^3)\||z_\rho|/\rho + |z|/\rho^2\|_{L^2}. \end{aligned}$$

In light of these estimates, and $\|z\|_X \ll 1$, Proposition 2.2 follows from the following lemma.

Lemma 2.3 *For $m \geq 3$ and $z(\rho)$ satisfying (2.16),*

1. $\|z\|_X \lesssim \|L_0 z\|_{L^2}$
2. $\left\||z_\rho| + \frac{|z|}{\rho}\right\|_{L^p} \lesssim \|L_0 z\|_{L^p}$ *for* $2 \leq p < \infty$
3. *if* $m > 3$, $\left\|\frac{|z_\rho|}{\rho} + \frac{|z|}{\rho^2}\right\|_{L^2} \lesssim \left\|\frac{L_0 z}{\rho}\right\|_{L^2}$.

Proof of the lemma. An estimate very similar to the first one here is proved in [11] (only the orthogonality condition is different). Here we prove the first and third statements together, by showing

$$\||z_\rho|/\rho^b + |z|/\rho^{1+b}\|_{L^2} \lesssim \|L_0 z/\rho^b\|_{L^2}$$

for $-1 \leq b \leq 1$. If this is false, we have a sequence $\{z_j\}$, with

$$\begin{aligned}
&\|(z_j)_\rho/\rho^b\|_{L^2}^2 + \|z_j/\rho^{1+b}\|_{L^2}^2 = 1, \\
&\int z_j(\rho) h_1(\rho) \rho d\rho = 0, \\
&\|L_0 z_j/\rho^b\|_{L^2} \to 0.
\end{aligned} \tag{2.25}$$

It follows that, up to subsequence, $z_j \to z^*$ weakly in H^1 and strongly in L^2 on compact subsets of $(0,\infty)$, and that $L_0 z^* = 0$. Hence $z^*(\rho) = Ch_1(\rho)$ for some $C \in \mathbb{C}$. Integration by parts gives

$$\|L_0 z_j/\rho^b\|_{L^2}^2 = \|(z_j)_\rho/\rho^b\|_{L^2}^2 + m\int_0^\infty \frac{|z_j|^2}{\rho^{2b+2}}(m + 2bh_3(\rho) - 2mh_1^2(\rho))\rho d\rho$$

and so, defining $V(\rho) := m + 2bh_3(\rho) - 2mh_1^2(\rho)$, we see that for any $\epsilon < 1/m$,

$$\limsup_{j\to\infty} m\int_0^\infty \frac{|z_j|^2}{\rho^{2b+2}}[V(\rho) - \epsilon]\rho d\rho \leq -m\epsilon.$$

If $2|b| + \epsilon < m$ (which certainly holds under our assumptions $|b| \leq 1$ and $m > 3$), then $\{\rho \mid V(\rho) - \epsilon \leq 0\}$ is a compact subset of $(0,\infty)$, and so

$$m\int_{V-\epsilon\leq 0} \frac{|C|^2 h_1^2(\rho)}{\rho^{2b+1}}[V(\rho) - \epsilon]\rho d\rho = \lim_{j\to\infty} m\int_{V-\epsilon\leq 0} \frac{|z_j|^2}{\rho^{2b+2}}[V(\rho) - \epsilon]\rho d\rho \leq -m\epsilon,$$

which implies $C \neq 0$. Finally, for any $\epsilon' > 0$,

$$\begin{aligned}
0 &= \lim_{j\to\infty} \int_0^\infty z_j(\rho) h_1(\rho)\rho d\rho \\
&= \int_{\epsilon'}^{1/\epsilon'} Ch_1^2(\rho)\rho d\rho + \lim_{j\to\infty}\left(\int_0^{\epsilon'} + \int_{1/\epsilon'}^\infty\right) z_j(\rho) h_1(\rho)\rho d\rho.
\end{aligned}$$

Since $\|z_j/\rho^{1+b}\|_{L^2} \leq 1$, and $\rho^{1+b} h_1 \in L^2$ (this is precisely where we need $m > 3$, for $b = 1$), the last integrals are uniformly small in ϵ', and we arrive at

$$0 = \int_0^\infty Ch_1^2(\rho)\rho d\rho,$$

contradicting $C \neq 0$.

We now prove the second statement. First note that following the proof of Lemma 4.4 in [11], the estimate

$$\| |z_\rho| + |z|/\rho \|_{L^p} \lesssim \|L_0 z\|_{L^p} + \|L_0 z\|_{L^2} \tag{2.26}$$

can be deduced from the X estimate above (the case $b = 0$). Now fix a smooth cut-off function $\Phi(t)$ with $\Phi(t) = 1$ for $t \in [0,1]$, $\Phi(t) = 0$ for $t \in [2,\infty)$, and $\Phi_t(t) < 0$ for $t \in (1,2)$. Let $\phi(\rho) := \Phi(t)$ with $t = (\rho/s)^\beta$, where $s \gg 1$ and $0 < \beta \ll 1$ are such that

$$\varepsilon_1 = \|\rho\phi_\rho(\rho)\|_{L^\infty} \lesssim \beta$$

and

$$\varepsilon_2 := \|\rho[1-\phi(\rho)]h_1(\rho)\|_{L^2(\rho d\rho)} \le \|\rho h_1(\rho)\|_{L^2((s,\infty),\rho d\rho)}$$

are sufficiently small. Now using (2.16),

$$|\int h_1 z\phi\rho d\rho| = |\int h_1 z(1-\phi)\rho d\rho| \le \varepsilon_2 \|z/\rho\|_{L^2(\rho d\rho)}$$

Observe that the proof of the X estimate above (and hence also of (2.26)), works even if $|\int h_1 z\rho d\rho| = o(1)\|z/\rho\|_{L^2}$, and so provided ε_2 is sufficiently small, we can apply (2.26) to obtain

$$\begin{aligned}
\|z/\rho\|_p &\le \|z\phi/\rho\|_p + \|z(1-\phi)/\rho\|_p \\
&\lesssim \|L_0(z\phi)\|_p + \|L_0(z\phi)\|_2 + \|z(1-\phi)/\rho\|_p \\
&\lesssim \|L_0(z\phi)\|_p + \|z(1-\phi)/\rho\|_p
\end{aligned}$$

Now $1-\phi$ is supported for $\rho \ge s \gg 1$, and on this set $h_3(\rho) \ge 1/2$. Then an easy adaptation of Lemma 4.2 in [11] (using $m > 1$) yields

$$\|z(1-\phi)/\rho\|_p \lesssim \|L_0(z(1-\phi))\|_p,$$

and hence

$$\begin{aligned}
\|z/\rho\|_p &\lesssim \|L_0(z\phi)\|_p + \|L_0(z(1-\phi))\|_p \\
&\lesssim \|L_0(z)\phi\|_p + \|L_0(z)(1-\phi)\|_p + \|z\phi_\rho\|_p.
\end{aligned}$$

Since $\|z\phi_\rho\|_p \le \varepsilon_1 \|z/\rho\|_p$, we conclude

$$\|z_\rho\|_p + \|z/\rho\|_p \le C\|L_0(z)\|_p + C\varepsilon_1\|z/\rho\|_p.$$

If ε_1 is small enough, the last term can be absorbed to the left side.

That completes the proof of the lemma, and hence of Proposition 2.2. □

Combining Proposition 2.2 with Lemma 2.1 leads to

Corollary 2.4 *Under the conditions of Proposition 2.2, if $m > 3$,*

$$|s^{-1}\dot{s}| + |\dot{\alpha}| \lesssim \|q/r\|_{L^2}^2. \tag{2.27}$$

This is our main estimate of the harmonic map parameters $s(t)$ and $\alpha(t)$.

2.5 Nonlinear estimates

We can now use Proposition 2.2 to estimate the nonlinear terms in (2.22). The idea is that from the splitting of Section 2.1, we expect $v_3(r,t) = h_3(r/s(t)) +$ "small". We will "freeze" the scaling factor $s(t)$ at, say, $s_0 := s(0)$ (and without loss of generality we will rescale the solution so that $s_0 = 1$) and treat the corresponding correction as a nonlinear term:

$$iq_t + \Delta_r q - \frac{1+m^2-2mh_3(r)}{r^2}q = Uq + Sq \tag{2.28}$$

where

$$U := \frac{1}{r^2}[m(v_3-h_3)(m(v_3+h_3)-2) + mr((v_3)_r - (h_3)_r)]$$

(here we have used $r(h_3)_r = mh_1^2$ and $h_1^2 + h_3^2 = 1$), and, recall from (2.24),

$$S(r,t) = -\frac{1}{2}Q(r,t) + \int_r^\infty \frac{1}{\tau}Q(\tau,t)d\tau, \qquad Q := |q|^2 + \frac{2m}{r}Re(\bar{\nu}q).$$

The next lemma estimates the r.h.s of (2.28) in various space-time norms.

Lemma 2.5 *Provided* (2.16) *holds, and* $\|z\|_X \ll 1$, *we have*

$$\|rUq\|_{L_t^2L_x^2} \lesssim ((1+\|s^{-1}\|_{L_t^\infty})\|s-1\|_{L_t^\infty} + \|q\|_{L_t^\infty L_x^2})\left\|\frac{q}{r}\right\|_{L_t^2L_x^2} + \|s^{-1}\|_{L_t^\infty}^{1/2}\|q\|_{L_t^4L_x^4}^2 \tag{2.29}$$

and

$$\|Sq\|_{L_t^{4/3}L_x^{4/3}} \lesssim \|q\|_{L_t^4L_x^4}(\|q\|_{L_t^4L_x^4}^2 + \|q/r\|_{L_t^2L_x^2}). \tag{2.30}$$

Proof. Recall

$$v_3(r) = h_3(r/s) + \xi_3(r/s) = (1+\gamma(r/s))h_3(r/s) + h_1(r/s)z_2(r/s),$$

and set, as usual, $\rho = r/s$. Estimate (2.29) follows from $\|z\|_{L^\infty} \lesssim \|z\|_X$, the estimates in Proposition 2.2, and

- $|h_3(r/s) - h_3(r)| = |\int_1^s \frac{d}{d\tau}h_3(r/\tau)d\tau| = m|\int_1^s \frac{1}{\tau}h_1^2(r/\tau)d\tau| \lesssim [\min(1,s)]^{-1}|s-1|$
- $r|[h_3(r/s)]_r - [h_3(r)]_r| = m|h_1^2(r/s) - h_1^2(r)| \lesssim [\min(1,s)]^{-1}|s-1|$.

For estimate (2.30), begin with

$$\|Sq\|_{L_x^{4/3}L_t^{4/3}} \le \|q\|_{L_t^4L_x^4}\|S\|_{L_t^2L_x^2}.$$

Using the Hardy-type inequality $\|\cdot\|_{L_x^2} \lesssim \|r\partial_r \cdot\|_{L_x^2}$ yields

$$\|S\|_{L_t^2L_x^2} \lesssim \|Q\|_{L_t^2L_x^2} \lesssim \|q\|_{L_t^4L_x^4}^2 + \|\nu\|_{L_t^\infty L_x^\infty}\left\|\frac{q}{r}\right\|_{L_t^2L_x^2}.$$

And since $|\nu| = |\hat{\mathbf{k}} - v_3\mathbf{v}| \lesssim 1$, we arrive at (2.30). $\square$

3 Dispersive estimates for critical-decay potentials in two dimensions

In order to establish any decay (dispersion) of solutions of (2.28), we need good dispersive estimates for the linear part

$$iq_t = -q_{rr} - \frac{1}{r}q_r + \frac{1}{r^2}(1 + m^2 - 2mh_3)q \tag{3.1}$$

This turns out to be a little tricky, since it is a "borderline" case in two senses: the space dimension is two, and the potential has $1/r^2$ behaviour both at the origin and at infinity, i.e.

$$\frac{1}{r^2}(1 + m^2 - 2mh_3(r)) \sim \begin{cases} \frac{(1+m)^2}{r^2} & r \to 0 \\ \frac{(1-m)^2}{r^2} & r \to \infty \end{cases} . \tag{3.2}$$

In this section we consider linear Schrödinger operators like the one appearing on the r.h.s of (3.1). More precisely, let

$$H = -\Delta + \frac{1}{r^2} + V(r), \qquad V \in C^\infty(0,\infty), \qquad 0 \le r^2 V(r) \le const. \tag{3.3}$$

Such an operator is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2 \backslash \{0\})$, extends to a self-adjoint operator on a domain $D(H)$ with $C_0^\infty(\mathbb{R}^2\backslash\{0\}) \subset D(H) \subset L^2(\mathbb{R}^2)$, and generates a one-parameter unitary group e^{-itH} such that for $\phi \in L^2$, $\psi = e^{-itH}\phi$ is the solution of the linear Schrödinger equation $i\psi_t = H\psi$ with initial data $\psi|_{t=0} = \phi$ (see, eg., [19]).

Our goal is to obtain dispersive space-time (*Strichartz*) estimates for e^{-itH} of the sort which hold for the "free" ($H = -\Delta$) evolution:

$$\|e^{it\Delta}\phi\|_{L^r_t L^p_x} + \left\| \int_0^t e^{i(t-s)\Delta} f(s) ds \right\|_{L^r_t L^p_x} \lesssim \|\phi\|_{L^2} + \|f\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}} \tag{3.4}$$

where (r,p) and $(\tilde{r},\tilde{p})$ are *admissible* pairs of exponents:

$$(r,p) \text{ admissible} \qquad \longleftrightarrow \qquad 1/r + 1/p = 1/2, \;\; 2 < r \le \infty,$$

and $p' = p/(p-1)$ denotes the Hölder dual exponent. The *endpoint* case of (3.4), $(r,p) = (2,\infty)$, is known to be false in general, but true for radial ϕ and f, save for the "double endpoint" case $r = \tilde{r} = 2$ ([24]).

Perturbative arguments to extend estimates like (3.4) to Schrödinger operators with potentials (in general one has to include a projection onto the continuous spectral subspace in order to avoid bound states, which do not disperse) cannot work for borderline behaviour like (3.2). Fortunately, the problem of obtaining dispersive estimates when the potential has this critical fall-off (and singularity) is taken up in a recent series of papers by Burq, Planchon, Stalker, and Tahvildar-Zadeh (see in particular [2, 3]). In place of a perturbative argument, the authors make a repulsivity assumption on the potential (which, in particular,

rule out bound states), and prove more-or-less directly – by identities – that solutions have some time decay, in a spatially-weighted space-time sense (a *Kato smoothing* - type estimate). This approach is ideally suited to our present problem: the operator appearing in (3.1) satisfies the following repulsivity property: when written in the form (3.3),

$$-r^2(rV(r))_r + 1 \geq \nu \quad \text{for some} \quad \nu > 0. \tag{3.5}$$

We cannot rely directly on the results of [2, 3] here. The paper [2] considers only potentials $(const)/r^2$, while the results of [3] hold in dimension ≥ 3 only, and do not immediately extend to dimension two for two reasons: one is the failure of the Hardy inequality, and the other is the failure of the double-endpoint Strichartz estimate (even for radial functions). However, we can recover the argument from [3] by exploiting the radial symmetry of our functions to avoid the Hardy inequality, and we can avoid the use of the double-endpoint Strichartz estimate by following the approach of [2], which in turn follows [21].

Theorem 3.1 *Suppose the Schrödinger operator H satisfies the conditions* (3.3) *and* (3.5). *Let $\phi = \phi(r)$ be radially symmetric. Then for any admissible pair (r, p), we have*

$$\|e^{-itH}\phi\|_{L_t^r L_x^p} + \left\| \frac{1}{|x|} e^{-itH}\phi \right\|_{L_t^2 L_x^2} \lesssim \|\phi\|_{L^2}. \tag{3.6}$$

If $f = f(r,t)$ is radially symmetric, and $(\tilde{r}, \tilde{p})$ is another admissible pair, then

$$\left\| \int_0^t e^{-i(t-s)H} f(x,s) ds \right\|_{L_t^r L_x^p} + \left\| \frac{1}{|x|} \int_0^t e^{-i(t-s)H} f(x,s) ds \right\|_{L_t^2 L_x^2} \lesssim \min\left(\|f\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}, \||x| f\|_{L_t^2 L_x^2} \right). \tag{3.7}$$

Remark 3.2 In [3], the single endpoint Strichartz estimate ((3.6) with $r = 2$) is also obtained for dimensions ≥ 3. In two dimensions, though it holds in the *free*, radial case, we do not know if it holds for our operators. However, it is *essential* to the present paper to have an estimate with L_t^2 decay (L_t^r with $r > 2$ is simply not enough – see the next section). Our way around this problem is to use the above *weighted* $L_t^2 L_x^2$ estimate that arises naturally in the approach of [3].

Proof. Parts of the proof are perturbative, so we identify a reference operator:

$$H = -\Delta + \frac{1}{r^2} + V =: H_0 + V.$$

Note that $H_0 = -\Delta + \frac{1}{r^2}$ satisfies the 'usual' Strichartz estimates (those satisfied by $-\Delta$ as in (3.4) above) on radial functions, since H_0 is simply $-\Delta$ conjugated by $e^{i\theta}$ when acting on such functions.

Step 1. Following [3], we begin with weighted resolvent estimates.

Lemma 3.3 *For $f = f(r)$ radial,*

$$\sup_{\mu \notin \mathbb{R}} \left\| \frac{1}{|x|}(H-\mu)^{-1} f \right\|_{L^2(\mathbb{R}^2)} \lesssim \||x| f\|_{L^2(\mathbb{R}^2)}. \tag{3.8}$$

Proof of Lemma. We can assume $f \in C_0^\infty(0,\infty)$, with the lemma then following from a standard density argument. Set $u := (H-\mu)^{-1} f$ so that $(H-\mu)u = f$, and note that $u = u(r)$ is radial, since f is. To avoid the use of the Hardy inequality in [3], we change variables from $u(r)$ to

$$v(x) := e^{i\theta} u(r)$$

and use $|\nabla v|^2 = |u_r|^2 + \frac{1}{r^2}|u|^2$, so

$$\left\| \frac{v}{|x|} \right\|_{L^2} \lesssim \|v\|_{H^1} \tag{3.9}$$

In terms of v, the equation for u becomes

$$(-\Delta + V - \mu)v = \tilde{f} \tag{3.10}$$

where $\tilde{f}(x) := e^{i\theta} f(r) \in L^2$, and so $v \in D(-\Delta + V) \subset H^2$. The proof of Lemma 3.3 now follows precisely the corresponding proof in [3], using $-d^2/d\theta^2 \geq 1$ on functions of our form $e^{i\theta} f(r)$, and with (3.9) (rather than Hardy) providing $v/|x| \in L^2$ where needed. □

Step 2. As in [3], the next step is to invoke [13] to conclude that the resolvent estimate (3.8) implies the following "Kato smoothing" weighted-L^2 estimate for the propagator: for $\phi = \phi(r)$,

$$\left\| \frac{1}{|x|} e^{-itH} \phi \right\|_{L_t^2 L_x^2} \lesssim \|\phi\|_{L^2}. \tag{3.11}$$

This is one part of (3.6). Note that the reference operator H_0 also satisfies the weighted estimate (3.11) (a fact which follows from the same argument). Another direct consequence of the resolvent estimate (3.8) is the inhomogeneous version of (3.11),

$$\left\| \frac{1}{|x|} \int_0^t e^{-i(t-s)H} f(\cdot, s) ds \right\|_{L_t^2 L_x^2} \lesssim \||x| f\|_{L_t^2 L_x^2}, \tag{3.12}$$

which is one part of (3.7). The estimate (3.7) is probably standard, but we did not see a proof, and so supply one in Section 6.2.

Step 3. Next we establish more of the inhomogeneous estimates in (3.7), but first for the reference operator H_0. Since we do not have the double-endpoint Strichartz estimate available, we now depart from [3] and henceforth follow [2] (which in turn relies partly on [21]). Note that by (3.11) for H_0, for any $\psi \in L_x^2$,

$$\begin{aligned}(\psi, \int_0^\infty e^{isH_0} f(\cdot, s) ds)_{L_x^2} &= \int_0^\infty ds (e^{-isH_0}\psi, f(\cdot, s))_{L_x^2} \\ &\leq \left\| \frac{1}{|x|} e^{-isH_0} \psi \right\|_{L_t^2 L_x^2} \||x| f\|_{L_t^2 L_x^2} \lesssim \|\psi\|_{L^2} \||x| f\|_{L_t^2 L_x^2},\end{aligned}$$

yielding

$$\left\| \int_0^\infty e^{isH_0} f(\cdot,s)ds \right\|_{L^2_x} \lesssim \| |x| f \|_{L^2_t L^2_x},$$

and hence by the Strichartz estimates for H_0, for (r,p) admissible,

$$\begin{aligned} \left\| \int_0^\infty e^{-i(t-s)H_0} f(\cdot,s)ds \right\|_{L^r_t L^p_x} &= \left\| e^{-itH_0} \int_0^\infty e^{isH_0} f(\cdot,s)ds \right\|_{L^r_t L^p_x} \\ &\lesssim \left\| \int_0^\infty e^{isH_0} f(\cdot,s)ds \right\|_{L^2_x} \lesssim \| |x| f \|_{L^2_t L^2_x}. \end{aligned}$$

Finally, the required estimate

$$\left\| \int_0^t e^{-i(t-s)H_0} f(\cdot,s)ds \right\|_{L^r_t L^p_x} \lesssim \| |x| f \|_{L^2_t L^2_x} \tag{3.13}$$

follows from a general argument of Christ-Kiselev ([6], and see [2]).

Step 4. To obtain the remaining part of (3.6) (the Strichartz estimate), we use (3.11), and (3.13), in a perturbative argument. We have

$$e^{-itH}\phi = e^{-itH_0}\phi + i\int_0^t e^{-i(t-s)H_0} V e^{-isH}\phi ds,$$

and so for (r,p) admissible,

$$\begin{aligned} \|e^{-itH}\phi\|_{L^r_t L^p_x} &\lesssim \|\phi\|_{L^2} + \| |x| V e^{-isH}\phi\|_{L^2_t L^2_x} \\ &\leq \|\phi\|_{L^2} + \| |x|^2 V\|_{L^\infty} \left\| \frac{1}{|x|} e^{-isH}\phi \right\|_{L^2_t L^2_x} \\ &\lesssim \|\phi\|_{L^2}. \end{aligned}$$

This finishes the proof of (3.6).

Step 5. It remains to prove the rest of the inhomogeneous estimates in (3.7). But given (3.6), these follow again from the argument used in Step 3.

That completes the proof of Theorem 3.1. □

Corollary 3.4 *If $m \geq 2$, the estimates* (3.6) *and* (3.7) *hold for the operator*

$$H := -\Delta + \frac{1}{r^2}(1 + m^2 - 2mh_3)$$

coming from the Schrödinger map problem.

Proof. We have

$$\frac{1}{r^2}(1+m^2-2mh_3) = \frac{1}{r^2} + V(r); \qquad V(r) = \frac{m}{r^2}(m-2h_3(r)).$$

So for $m \geq 2$,

$$(m+1)^2 \geq 1 + r^2V(r) \geq (m-1)^2 \geq 1,$$

and

$$1 - r^2(rV)_r = 1 + m(m - 2h_3(r) + 2mh_1^2(r)) \geq 1 + m(m-2) \geq 1.$$

Thus the conditions (3.3) and (3.5) both hold with $\nu = 1$. □

4 Proof of the main theorem

Let $\mathbf{u} \in C([0, T_{max}); \Sigma_m)$ be the solution of the Schrödinger map equation (1.1) with initial data $\mathbf{u}_0$ (given by Theorem 1.4). Energy is conserved:

$$\mathcal{E}(\mathbf{u}(t)) = \mathcal{E}(\mathbf{u}_0) = 4\pi m + \delta_1^2.$$

We begin by splitting the initial data $\mathbf{u}(0)$, using the following lemma, which is proved in Section 6.3:

Lemma 4.1 *If $m \geq 3$, and if δ is sufficiently small, then for any map $\mathbf{u} \in \Sigma_m$ with $\mathcal{E}(\mathbf{u}) \leq 4\pi m + \delta^2$, there exist $s > 0$, $\alpha \in \mathbb{R}$, and a complex function $z(\rho)$ such that*

$$\mathbf{u}(r,\theta) = e^{[m\theta+\alpha]R}[(1+\gamma(r/s))\mathbf{h}(r/s) + \mathbf{V}^{r/s}(z(r/s))] \tag{4.1}$$

with z satisfying (2.16)*; i.e.,*

$$\int_0^\infty z(\rho)h_1(\rho)\rho d\rho = 0, \tag{4.2}$$

and $\|z\|_X^2 \lesssim \mathcal{E}(\mathbf{u}) - 4\pi m$.

Invoking the lemma, we have

$$\mathbf{u}_0 = e^{[m\theta+\alpha_0]R}[(1+\gamma_0(r/s_0))\mathbf{h}(r/s_0) + \mathbf{V}^{r/s_0}(z_0(r/s_0))]$$

with z_0 satisfying the orthogonality condition (2.16), and

$$\|z_0\|_X \lesssim \delta_1 \ll 1.$$

Now rescale, setting

$$\hat{\mathbf{u}}(x,t) := \mathbf{u}(s_0 x, s_0^2 t).$$

Then $\hat{\mathbf{u}}$ is another solution of the Schrödinger map equation (1.1), and

$$\hat{\mathbf{u}}(x,0) = e^{[m\theta+\alpha_0]R}[(1+\gamma_0(r))\mathbf{h}(r) + \mathbf{V}^r(z_0(r))].$$

Let $q(r,t)$ be the complex function derived from the Schrödinger map $\hat{\mathbf{u}}$, as in Section 2.4.

Suppose (r,p) is an admissible pair of exponents. Define a spacetime norm Y by

$$\|q\|_Y := \|q\|_{L_t^\infty L_x^2 \cap L_t^4 L_x^4 \cap L_t^r L_x^p} + \left\|\frac{q}{r}\right\|_{L_t^2 L_x^2}.$$

As long as $\|z\|_X \lesssim \|q\|_{L_x^2}$ remains sufficiently small, Corollary 3.4 together with estimates (2.29)- (2.30) yields

$$\|q\|_Y \lesssim \|q(0)\|_{L^2} + \Big[(1+\|s^{-1}\|_{L_t^\infty})\|s-1\|_{L_t^\infty} + (1+\|s^{-1}\|_{L_t^\infty})\|q\|_Y + \|q\|_Y^2\Big]\|q\|_Y. \quad (4.3)$$

We also have

$$\hat{\mathbf{u}} = e^{[m\theta+\alpha(t)]R}[(1+\gamma(r/s(t),t))\mathbf{h}(r/s(t)) + \mathbf{V}^{r/s(t)}(z(r/s(t),t))],$$

with $z(\rho,t)$ satisfying (2.16), $s(0)=1$, $\alpha(0)=\alpha_0$, and, by Corollary 2.4, $s(t) \in C([0,T);\mathbb{R}^+)$ and $\alpha(t) \in C([0,T);\mathbb{R})$, with

$$\|s^{-1}\dot{s}\|_{L_t^1} + \|\dot{\alpha}\|_{L_t^1} \lesssim \|q\|_Y^2. \quad (4.4)$$

Taking $\|q(0)\|_{L^2} \lesssim \delta_1$ sufficiently small, the estimates (4.3) and (4.4) yield

$$\|q\|_Y \lesssim \delta_1, \qquad \|s^{-1}\dot{s}\|_{L_t^1} + \|\dot{\alpha}\|_{L_t^1} \lesssim \delta_1^2 \quad (4.5)$$

(and in particular, $\|z\|_X \ll 1$ continues to hold). Since

$$|\nabla[\hat{\mathbf{u}} - e^{[m\theta+\alpha(t)]R}\mathbf{h}(r/s(t))]| \lesssim \frac{1}{s}(|z_\rho| + |z/\rho|)(1+|z|),$$

the estimates of Proposition 2.2 give

$$\|\nabla[\hat{\mathbf{u}} - e^{[m\theta+\alpha(t)]R}\mathbf{h}(r/s(t))]\|_Y \lesssim \|q\|_Y \lesssim \delta_1. \quad (4.6)$$

Estimate (4.5) shows: (a) that $s(t) \geq \mathit{const} > 0$, and hence, by Corollary 1.5, we must have $T_{max} = \infty$; (b) that

$$s(t) \to s_\infty \in (1 - c\delta_1^2, 1 + c\delta_1^2), \qquad \alpha(t) \to \alpha_\infty \in (\alpha_0 - c\delta_1^2, \alpha_0 + c\delta_1^2)$$

as $t \to \infty$.

Finally, undoing the rescaling, $\mathbf{u}(r,t) = \hat{\mathbf{u}}(r/s_0, t/s_0^2)$, yields the estimates of Theorem 1.7. □

5 Appendix: local wellposedness

In this appendix we prove Theorem 1.4 and Corollary 1.5 on the local wellposedness of the Schrödinger flow (1.1) when the data $\mathbf{u}_0 \in \Sigma_m$ has energy $\mathcal{E}(\mathbf{u}_0) = 4\pi m + \delta_0^2$ close to the harmonic map energy, $0 < \delta_0 \le \delta \ll 1$. In subsection 5.1 we show that z (and hence $\mathbf{u}$) can be reconstructed from q, s, and α. This subsection is time-independent. In subsection 5.2 we set up the equations for the existence proof. In subsection 5.3 we show that we have a contraction mapping, and complete the proof of Theorem 1.4 and Corollary 1.5. In subsection 5.4 we discuss the small energy case.

Recall the decomposition $\mathbf{u}(r,\theta) = e^{m\theta R}\mathbf{v}(r)$ and

$$\mathbf{v}(r) = e^{\alpha R}\left[\mathbf{h}(\rho) + \boldsymbol{\xi}(\rho)\right] = e^{\alpha R}\begin{bmatrix}(1+\gamma)h_1 - h_3 z_2\\ z_1\\ (1+\gamma)h_3 + h_1 z_2\end{bmatrix}(\rho), \tag{5.1}$$

where $\rho = r/s$, $\boldsymbol{\xi} = z_1\widehat{\boldsymbol{\jmath}} + z_2 h \times \widehat{\boldsymbol{\jmath}} + \gamma b h$ and $\gamma = \sqrt{1-|z|^2} - 1$. The time-dependence of $\mathbf{u}, \mathbf{v}, \boldsymbol{\xi}, \alpha, s$ and γ has been dropped from (5.1). The equation $D_r\mathbf{e} = 0$ is equivalent to

$$\mathbf{e}_r = -(\mathbf{v}_r \cdot \mathbf{e})\mathbf{v}. \tag{5.2}$$

Recall $q\mathbf{e} = \mathbf{v}_r - \frac{m}{r}J^{\mathbf{v}}R\mathbf{v}$ with $\nu\mathbf{e} = J^{\mathbf{v}}R\mathbf{v} = \widehat{\mathbf{k}} - v_3\mathbf{v}$. By substituting in (5.1) and using $L_0\mathbf{h} = \frac{m}{r}\widehat{\mathbf{k}}$, $q\mathbf{e}$ should satisfy

$$se^{-\alpha R}q\mathbf{e}(r) = (L_0 z)(\rho)\widehat{\boldsymbol{\jmath}} + \mathbf{G_0}(z)(\rho), \quad \rho = r/s, \tag{5.3}$$

where

$$\mathbf{G_0}(z)(\rho) := se^{-\alpha R}[\mathbf{v}_r - \frac{m}{r}(\widehat{\mathbf{k}} - v_3\mathbf{v})] - (L_0 z)\widehat{\boldsymbol{\jmath}} = \gamma_\rho\mathbf{h} + \frac{m}{\rho}(\gamma\widehat{\mathbf{k}} + \gamma h_3\mathbf{h} + \xi_3\boldsymbol{\xi}) \tag{5.4}$$

and $\|\mathbf{G_0}(z)\|_{L^2} \lesssim \|z\|_X^2$ when $\|z\|_X \ll 1$. In other words, q is rescaled $L_0 z$, plus error.

In this Appendix, we will choose a different orthogonality condition for z, instead of (2.16). Specifically, we choose the unique s and α so that

$$\langle h_1, z\rangle_X = 0. \tag{5.5}$$

(Recall $\langle f, g\rangle_X = \int_0^\infty (\bar{f}_r g_r + \frac{m^2}{r^2}\bar{f}g)rdr$.) The condition (5.5) makes sense for all $m \neq 0$ and suffices for the proof of local wellposedness. In contrast, (2.16) makes sense only if $|m| \ge 3$, but is necessary for the study of the time-asymptotic behavior. In [11, Sect. 2], we chose s and α to minimize $\left\|\mathbf{u} - e^{(m\theta+\alpha)R}\mathbf{h}(\cdot/s)\right\|_{\dot{H}^1}$. The resulting equations in [11, Lem. 2.6] are $\langle h_1, z_1\rangle_X = 0$ and $\langle h_1, z_2\rangle_X = \int_0^\infty \frac{4m^2}{\rho^2}h_1^2 h_3\gamma(\rho)\rho d\rho$. The condition (5.5) is similar but has no error term. The unique choice of s and α can be proved by implicit function theorem, similar to the proof for Lemma 4.1, and is skipped. It is important to point out, however, that the parameter s used here, though not the same as $s(\mathbf{u})$ defined in (1.20)-(1.21), is nonetheless comparable: $s = s(\mathbf{u})(1 + O(\delta_0^2))$ (this comes immediately from the implicit function theorem argument). Thus we can state the local well-posedness result (Theorem 1.4) in terms of $s(\mathbf{u}_0)$.

5.1 Reconstruction of z and $\mathbf{u}$ from q, s, and α

In this subsection all maps are time-independent. For a given map $\mathbf{u} = e^{m\theta R}\mathbf{v}(r) \in \Sigma_m$ with energy close to $4\pi m$, we can define s, α, z and q. The three quantities s, α, z determine $\mathbf{u}$, and hence q. Conversely, as will be done in Lemma 5.2 of this subsection, we can recover z and $\mathbf{u}$ if s, α and q are given, assuming that $\|q\|_{L^2} \le \delta$. Before that we first prove difference estimates for $\delta\mathbf{e}$ in Lemma 5.1.

For given $s > 0$, $\alpha \in \mathbb{R}$ and $z \in X$ small, we define $\mathbf{v}(r) = \mathbf{V}(z, s, \alpha)(r)$ by (5.1), and $\mathbf{e}(r) = \hat{\mathbf{E}}(z, s, \alpha)(r)$ by the ODE

$$\mathbf{e}(z)(0) = e^{\alpha R}\hat{\boldsymbol{\jmath}}, \quad \mathbf{e}_r = -(\mathbf{v}_r \cdot \mathbf{e})\mathbf{v}, \quad \text{where } \mathbf{v} = \mathbf{V}(z, s, \alpha). \tag{5.6}$$

Also denote $\hat{\mathbf{E}}(z) = \hat{\mathbf{E}}(z, 1, 0)$. Simple comparison shows

$$\hat{\mathbf{E}}(z, s, \alpha) = e^{\alpha R}\hat{\mathbf{E}}(z^s), \quad z^s(r) := z(r/s). \tag{5.7}$$

Lemma 5.1 *Suppose $z_l \in X$, $l = a, b$, are given with $\|z_l\|_X$ sufficiently small. Let $\delta z := z_a - z_b$, $\delta\mathbf{v} := \mathbf{V}(z_a, 1, 0) - \mathbf{V}(z_b, 1, 0)$, and $\delta\mathbf{e} := \hat{\mathbf{E}}(z_a) - \hat{\mathbf{E}}(z_b)$. Then*

$$\|\delta\mathbf{v}\|_X + \|\delta\mathbf{e}\|_{L^\infty} \lesssim \|\delta z\|_X .$$

Proof. Note

$$\|\mathbf{h}_r\|_{L^2(rdr)} \le C; \quad \|\boldsymbol{\xi}^l\|_X \lesssim \|z^l\|_X + \|z^l\|_X^2, \quad l = a, b. \tag{5.8}$$

Since $\delta\mathbf{v} = \delta\boldsymbol{\xi} = (\delta z)\hat{\boldsymbol{\jmath}} + (\delta\gamma)\mathbf{h}$,

$$\|\delta\mathbf{v}\|_X + \|\delta\boldsymbol{\xi}\|_X \lesssim (1 + \|z_a\|_X + \|z_b\|_X)\|\delta z\|_X \lesssim \|\delta z\|_X. \tag{5.9}$$

For $\delta\mathbf{e}$, write $\delta\mathbf{e} = (\delta e_1, \delta e_2, \delta e_3)$ and

$$\delta e_{j,r} = -(\delta\boldsymbol{\xi}_r \cdot e_a)v_{a,j} - (\mathbf{v}_{b,r} \cdot \delta\mathbf{e})v_{a,j} - (\mathbf{v}_{b,r} \cdot \mathbf{e}_b)\delta\boldsymbol{\xi}_j, \quad j = 1, 2, 3. \tag{5.10}$$

First consider δe_2. Integrate in r. Using (5.8), (5.9), $v_{a,2} = z_{a,1}$, and $v_{l,r} \in L^2(rdr)$,

$$\begin{aligned} |\delta e_2(\tau)| &\lesssim \int_0^\tau \left(|\delta\boldsymbol{\xi}_r \cdot \mathbf{e}_a)\frac{z_a}{r}| + |(\mathbf{v}_{b,r} \cdot \delta\mathbf{e})\frac{z_a}{r}| + |(\mathbf{v}_{b,r} \cdot \mathbf{e}_b)\frac{\delta z_1}{r}|\right) rdr \\ &\lesssim (1 + \max_{l=a,b}\|z_l\|_X)\,\|\delta z\|_X + \max_{l=a,b}\|z_l\|_X\,\|\delta\mathbf{e}\|_{L^\infty} . \end{aligned} \tag{5.11}$$

Next we consider δe_1 and δe_3. Equations (5.10) for $j = 1, 3$ can be written as a vector equation for $x = (\delta e_1, \delta e_3)^T$:

$$x_r = A(r)x + F, \tag{5.12}$$

where

$$A(r) = -\begin{bmatrix} h_1 \\ h_3 \end{bmatrix}[h_{1,r}, h_{3,r}] = \frac{m}{r}h_1\begin{bmatrix} h_1h_3, & -h_1^2 \\ h_3^2, & -h_1h_3 \end{bmatrix}$$

and

$$F = \begin{bmatrix} F_1 \\ F_3 \end{bmatrix}, \quad F_j = -(\delta\boldsymbol{\xi}_r \cdot \mathbf{e}_a) v_{a,j} - (\boldsymbol{\xi}_{b,r} \cdot \delta\mathbf{e}) h_j - (\mathbf{v}_{b,r} \cdot \delta\mathbf{e}) \xi_{a,j} - (\mathbf{v}_{b,r} \cdot \mathbf{e}_b) \delta\xi_j, \quad j = 1, 3.$$

To simplify the linear part $\tilde{x}_r = A(r)\tilde{x}$, let $y = U^{-1}\tilde{x}$ where

$$U(r) = \begin{bmatrix} h_1, & -h_3 \\ h_3, & h_1 \end{bmatrix}, \quad U^{-1} = \begin{bmatrix} h_1, & h_3 \\ -h_3, & h_1 \end{bmatrix}.$$

Then y satisfies

$$y_r = (U^{-1})_r U y + U^{-1} A U y = \frac{m}{r} h_1 \begin{bmatrix} 0, & 0 \\ -1, & 0 \end{bmatrix} y.$$

This linear system can be solved explicitly,

$$y(r) = \begin{bmatrix} 1, & 0 \\ p(\rho, r), & 1 \end{bmatrix} y(\rho), \quad p(\rho, r) = -\left(\int_\rho^r \frac{m}{r} h_1(\tau) d\tau \right) = -\big[2 \arctan \tau^m \big]_\rho^r.$$

Thus the linear system $\tilde{x}_r = A(r)\tilde{x}$ has the solution $\tilde{x}(r) = P(\rho, r)\tilde{x}(\rho)$ with the propagator

$$P(\rho, r) = U(r) \begin{bmatrix} 1, & 0 \\ p(\rho, r), & 1 \end{bmatrix} U^{-1}(\rho).$$

The original system (5.12) with $x(0) = 0$ has the solution

$$x(r) = \int_0^r P(\rho, r) F(\rho) d\rho.$$

To estimate $x(r)$, the two terms of F_3 with h_3 as the last factor,

$$\tilde{F}_3 = -(\delta\boldsymbol{\xi}_r \cdot \mathbf{e}_a) h_3 - (\boldsymbol{\xi}_{b,r} \cdot \delta\mathbf{e}) h_3$$

require special care since it may not be in $L^1(dr)$. Other terms can be estimated as follows:

$$\Big| \int_0^r P(\rho, r) \begin{bmatrix} F_1 \\ F_3 - \tilde{F}_3 \end{bmatrix} d\rho \Big| \lesssim \int_0^\infty |F_1| + |F_3 - \tilde{F}_3| dr \lesssim \|\delta z\|_X + (\|z_a\|_X + \|z_b\|_X) \|\delta\mathbf{e}\|_{L^\infty}.$$

We treat $\tilde{F}_3$ by integration by parts:

$$\int_0^r P(\rho, r) \begin{bmatrix} 0 \\ \tilde{F}_3 \end{bmatrix} d\rho = \int_0^r P(\rho, r) \begin{bmatrix} 0 \\ -(\delta\boldsymbol{\xi}_\rho \cdot \mathbf{e}_a + \boldsymbol{\xi}_{b,\rho} \cdot \delta\mathbf{e}) h_3 \end{bmatrix} d\rho$$

$$= - \begin{bmatrix} 0 \\ (\delta\boldsymbol{\xi} \cdot \mathbf{e}_a + \boldsymbol{\xi}_b \cdot \delta\mathbf{e}) h_3 \end{bmatrix} (r) + \int_0^r P(\rho, r) \begin{bmatrix} 0 \\ (\delta\boldsymbol{\xi} \cdot \mathbf{e}_{a,\rho} + \boldsymbol{\xi}_b \cdot \delta e_\rho) h_3 + (\delta\boldsymbol{\xi} \cdot \mathbf{e}_a + \boldsymbol{\xi}_b \cdot \delta\mathbf{e}) h_{3,\rho} \end{bmatrix} d\rho$$

$$+ \int_0^r P_\rho(\rho, r) \begin{bmatrix} 0 \\ (\delta\boldsymbol{\xi} \cdot \mathbf{e}_a) h_3 + (\boldsymbol{\xi}_b \cdot \delta\mathbf{e}) h_3 \end{bmatrix} d\rho = \sum_{j=1}^3 I_j.$$

Now we estimate the right side one by one. For I_1,

$$|I_1| \lesssim \|\delta\boldsymbol{\xi}\|_{L^\infty} + \|\boldsymbol{\xi}_b\|_{L^\infty}\|\delta\mathbf{e}\|_{L^\infty} \lesssim \|\delta z\|_X + \|z_b\|_X \|\delta\mathbf{e}\|_{L^\infty}.$$

For I_2, observe that

$$\|\mathbf{e}_{a,r}\|_{L^2} \le C, \quad \|\delta\mathbf{e}_r\|_{L^2} \lesssim \|\delta z\|_X + \|\delta\mathbf{e}\|_{L^\infty},$$

due to the facts that $\mathbf{e}_{a,r} = -(\mathbf{v}_{a,r}\cdot\mathbf{e}_a)\mathbf{v}_a$ and $\delta\mathbf{e}_r = -(\mathbf{v}_{a,r}\cdot\mathbf{e}_a)\mathbf{v}_a + (\mathbf{v}_{b,r}\cdot\mathbf{e}_b)\mathbf{v}_b$. Thus

$$|I_2| \lesssim \|\delta z\|_X + \|z_b\|_X (\|\delta z\|_X + \|\delta\mathbf{e}\|_{L^\infty}).$$

To estimate the last term I_3, note that

$$P_\rho(\rho, r) = \frac{m}{\rho} h_1(\rho) U(r) \cdot \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot U^{-1}(\rho) + \begin{bmatrix} 1 & 0 \\ p(\rho, r) & 1 \end{bmatrix} \cdot \begin{bmatrix} -h_3, & h_1 \\ -h_1, & -h_3 \end{bmatrix} (\rho) \right\}$$

and hence $|P_\rho(\rho, r)| \lesssim h_1(\rho)/\rho$. We get

$$|I_3| \lesssim \int_0^r \left|\frac{h_1}{\rho}\right| (|(\delta\boldsymbol{\xi}\cdot\mathbf{e}_a)h_3| + |(\boldsymbol{\xi}_b\cdot\delta\mathbf{e})h_3|)\, d\rho \lesssim \|\frac{h_1}{\rho}\|_{L^2}(\|\delta z\|_X + \|z_b\|_X \|\delta\mathbf{e}\|_{L^\infty}).$$

Summing up, we have shown

$$\|\delta\mathbf{e}\|_{L^\infty} \lesssim \|\delta z\|_X + (\|z_a\|_X + \|z_b\|_X)\|\delta\mathbf{e}\|_{L^\infty}.$$

Since $\|z_a\|_X + \|z_b\|_X \ll 1$, we can absorb the last term to the left side. The lemma is proved. $\square$

Lemma 5.2 *For given $s > 0$, $\alpha \in \mathbb{R}$, and $q \in L^2_{rad}$ with $\|q\|_{L^2} \le \delta$, there is a unique function $z = Z(q, s, \alpha) \in X$ so that $\langle h_1, z\rangle_X = 0$, $\|z\|_X \lesssim \delta$ and the function $\mathbf{v} = \mathbf{V}(Z, s, \alpha)$ satisfies (5.3). Moreover, $Z(q, s, \alpha)$ is independent of α and continuous in q and s.*

Proof. Simple comparison shows

$$Z(q, s, \alpha) = Z(q(\cdot s), 1, 0). \tag{5.13}$$

Thus it suffices to prove the case $s = 1$ and $\alpha = 0$. We will construct $Z(q, 1, 0)$ by a contraction mapping argument. Define the map

$$\Phi^q(z)(r) = L_0^{-1}\Pi[q\hat{\mathbf{E}}(z) - \mathbf{G_0}(z)](r), \tag{5.14}$$

where $\Pi = (\mathbf{V}^r)^{-1}P^{h(r)}$ is a projection of vector fields on $\mathbb{R}^+$ to $L^2(rdr)$, with the mapping $(\mathbf{V}^r)^{-1} : T_{\mathbf{h}(r)}\mathbb{S}^2 \to \mathbb{C}$ and the projection $P^{\mathbf{h}(r)} : \mathbb{R}^3 \to T_{\mathbf{h}(r)}\mathbb{S}^2$ defined in Section 2.2; L_0^{-1} is the inverse map of L_0 and maps $L^2(rdr)$ to the X-subspace $h_1^\perp$; $\hat{\mathbf{E}}(z)$ is defined after (5.6), and $\mathbf{G_0}(z)$ is defined by (5.4).

We will show that Φ^q is a contraction mapping in the class

$$\mathcal{A}_\delta = \{z \in X : \|z\|_X \le 2C_1\delta\}\,, \quad C_1 = \|L_0^{-1}(\mathbf{V}^r)^{-1}P^{\mathbf{h}(r)}\|_{B(L^2,X)}$$

for sufficiently small $\delta > 0$. First,

$$\|\Phi^q(z)\|_X \le C_1 \|q\|_2 + C\|\mathbf{G_0}\|_2 \le C_1\delta + C\|z\|_X^2\,.$$

Thus Φ^q maps $\mathcal{A}$ into itself if δ is sufficiently small. We now prove difference estimates for Φ^q. Suppose $z_a, z_b \in \mathcal{A}$ are given and let $\mathbf{v}_l = \mathbf{V}(z_l)$ and $\mathbf{e}_l = \hat{\mathbf{E}}(z_l)$, $l = a, b$. Also define $\boldsymbol{\xi}_l$ by (5.1) and note $\delta\boldsymbol{\xi} = \delta\mathbf{v}$. By Lemma 5.1,

$$\|\delta\mathbf{v}\|_X + \|\delta\boldsymbol{\xi}\|_X + \|\delta\mathbf{e}\|_{L^\infty} \lesssim \|\delta z\|_X\,.$$

We now estimate $\delta\mathbf{G_0}(z) = \mathbf{G_0}(z_a) - \mathbf{G_0}(z_b)$ in L_p, $p = 2, 4$ (we need $p = 4$ later):

$$\|\delta\mathbf{G_0}(z)\|_{L_p} \lesssim \|\delta\gamma_r\|_{L_p} + \|\frac{\delta\gamma}{r}\|_{L_p} + \|\delta(\xi_3\boldsymbol{\xi})\|_{L_p} \lesssim (\|z_a\|_X + \|z_b\|_X)\,\|\delta z\|_{X_p}\,. \tag{5.15}$$

Thus,

$$\|\Phi^q(z_a) - \Phi^q(z_b)\|_X \lesssim \|q\delta\mathbf{e} - \delta\mathbf{G_0}(z)\|_{L^2} \lesssim \|q\|_{L^2}\|\delta\mathbf{e}\|_{L^\infty} + (\|z_a\|_X + \|z_b\|_X)\,\|\delta z\|_X \ll \|\delta z\|_X\,. \tag{5.16}$$

Thus Φ^q is indeed a contraction mapping and the function $Z(q, s, \alpha)$ exists.

We now consider the continuity. The continuity in s follows from (5.13), although it may not be Hölder continuous. For the continuity in q, let q_a and q_b be given and $z_l = Z(q_l, s, \alpha)$, $l = a, b$. An estimate similar to (5.16) shows

$$\|\delta z\|_X = \|\Phi^{q_a}(z_a) - \Phi^{q_b}(z_b)\|_X \lesssim \|\delta q\|_{L^2} + \varepsilon\|\delta z\|_X\,, \tag{5.17}$$

where $\varepsilon = \|q_a\|_{L^2} + \|q_b\|_{L^2} + \|z_a\|_X + \|z_b\|_X \ll 1$ and hence $\varepsilon\|\delta z\|_X$ can be absorbed to the left side. This shows continuity in q in L^2-norm. $\square$

5.2 Evolution system of q, s and α

By (5.1), the dynamics of $\mathbf{u}$ is completely determined by the dynamics of z, s and α. Because of Lemma 5.2, it is also completely determined by the dynamics of q, s and α. The latter system is preferred by us since the q equation is easier than the z equation to estimate, and q lies in a more familiar space L^2, rather than z in X.

The equations for z and q are given by (2.12) and (2.22), respectively. However, since we choose the orthogonality condition (5.5), i.e., $\langle h_1, z(t)\rangle = 0$ for all t, the equations for s and α are different from (2.18).

We now specify the equations we will use. Let $\tilde{q} := e^{i(m+1)\theta}q$. Recall $\nu\mathbf{e} = \nu_1\mathbf{e} + \nu_2 J^v\mathbf{e} = J^{\mathbf{v}}R\mathbf{v} = \widehat{\mathbf{k}} - v_3\mathbf{v}$ and $\nu_r = v_3(q + \frac{m}{r}\nu)$. By (2.22) and an integration by parts on the potential defined in (2.23), we obtain

$$i\tilde{q}_t + \Delta\tilde{q} = V\tilde{q}, \quad V = V_1 - V_2 + \int_r^\infty \frac{2}{r'}V_2(r')dr' \tag{5.18}$$

where

$$V_1 := \frac{m(1+v_3)(mv_3-m-2)}{r^2} + \frac{mv_{3,r}}{r}, \quad V_2 := \frac{1}{2}|q|^2 + \operatorname{Re}\frac{m}{r}\bar{\nu}q.$$

For s and α, condition (5.5) implies $\langle h_1, \partial_t z(t)\rangle_X = 0$. Substituting in (2.12), we get

$$\left\langle h_1,\ \left(s^2\dot{\alpha} - ims\dot{s}\right)(1+\gamma)h_1 + s^2\dot{\alpha}izh_3 - s\dot{s}rz_r\right\rangle_X = \langle h_1,\ -iNz + P\mathbf{F_1}\rangle_X\,. \tag{5.19}$$

Note

$$\langle h_1,\ Nz\rangle_X = (L_0N_0h_1, L_0z)_{L^2}, \quad \langle h_1,\ r\partial_r z\rangle_X = (rN_0h_1, z_r)_{L^2}.$$

Let $G_1 := \langle h_1, P\mathbf{F_1}\rangle_X = (N_0h_1, P\mathbf{F_1})_{L^2}$ where $N_0 := -\Delta_r + \frac{m^2}{r^2}$. By Lemma 6.1 with $g = N_0h_1$,

$$G_1 = \int_0^\infty \Big(ig_r(-\gamma z_r + z\gamma_r) + \frac{m}{r}h_1 g(-2\gamma_r - iz_2z_{1,r} + iz_1z_{2,r}) \\ + \frac{m}{r}(h_1g)_r(\gamma^2 - iz_2z) + i\frac{m^2}{r^2}(2h_1^2-1)g\gamma z\Big) rdr.$$

Separating real and imaginary parts, we can rewrite (5.19) as a system for $\dot{\alpha}$ and $\dot{s}$:

$$\left(\|h_1\|_X^2\, I + A\right)\begin{bmatrix} s^2\dot{\alpha} \\ -ms\dot{s}\end{bmatrix} = \vec{G}_2 := \begin{bmatrix} (L_0N_0h_1, L_0z_2)_2 \\ -(L_0N_0h_1, L_0z_1)_2\end{bmatrix} + \begin{bmatrix}\operatorname{Re} G_1 \\ \operatorname{Im} G_1\end{bmatrix}, \tag{5.20}$$

where

$$I = \begin{bmatrix} 1, & 0 \\ 0, & 1\end{bmatrix}, \quad A = \begin{bmatrix} \langle h_1,\, \gamma h_1 - z_2h_3\rangle_X\,, & \frac{1}{m}(rN_0h_1,\, z_{1,r})_{L^2} \\ \langle h_1,\, z_1h_3\rangle_X\,, & \langle h_1,\, \gamma h_1\rangle_X + \frac{1}{m}(rN_0h_1, z_{2,r})_{L^2}\end{bmatrix}.$$

We have $\|A\|_{L^\infty} \lesssim \|z\|_X$.

We will study the integral equation version of (5.18) and (5.20) for $\tilde{q}$, s, and α:

$$\tilde{q}(t) = e^{-it\Delta}\tilde{q}_0 - i\int_0^t e^{-i(t-\tau)\Delta}(V\tilde{q})(\tau)\,d\tau, \tag{5.21}$$

$$\begin{bmatrix} s(t) \\ \alpha(t)\end{bmatrix} = \begin{bmatrix} s_0 \\ \alpha_0\end{bmatrix} + \int_0^t \left\{ \begin{bmatrix} 0 & -(ms)^{-1} \\ s^{-2} & 0\end{bmatrix}\left(\|h_1\|_X^2\, I + A\right)^{-1}\vec{G}_2\right\}(\tau)\,d\tau. \tag{5.22}$$

5.3 Contraction mapping and conclusion

Let $q_0 \in L^2_{rad}(\mathbb{R}^2)$, $s_0 > 0$, and $\alpha_0 \in \mathbb{R}$ be given, with $\|q_0\|_{L^2} \le \delta$. For $\delta, \sigma > 0$ sufficiently small we will find a solution of (5.21)–(5.22) for $t \in I = [0, \sigma s_0^2]$.

We will first construct the solution assuming $s_0 = 1$. The solution for general s_0 is obtained from rescaling,

$$\mathbf{u}(t,x) = \tilde{\mathbf{u}}(t/s_0^2, x/s_0)$$

where $\tilde{u}$ is the solution corresponding to initial data $\tilde{\mathbf{u}}_0(x) = \mathbf{u}_0(x/s_0)$, and $s(\tilde{\mathbf{u}}_0) = 1$.

Assuming $s_0 = 1$, we will define a (contraction) mapping in the following class

$$\mathcal{A}_{\delta,\sigma} = \big\{(\tilde{q}, s, \alpha) : I = [0,\sigma] \to L^2(\mathbb{R}^2) \times \mathbb{R}^+ \times \mathbb{R} : \\ \|\tilde{q}\|_{Str[I]} \le \delta;\ \forall t,\ q(t) = e^{-i(m+1)\theta}\tilde{q}(t) \in L^2_{rad},\ s(t) \in [0.5, 1.5]\big\}, \quad (5.23)$$

for sufficiently small $\delta, \sigma > 0$. Here

$$\|q\|_{Str[I]} \equiv \|q\|_{L^\infty_t L^2_x[I] \cap L^4_t L^4_x[I] \cap L^{8/3}_t L^8_x[I]} .$$

The map is defined as follows. Let $\tilde{q}_0 = e^{i(m+1)\theta} q_0$. Suppose $Q = (\tilde{q}, s, \alpha)(t) \in \mathcal{A}_{\delta,\sigma}$ has been chosen. For each $t \in I$, let $q = e^{-i(m+1)\theta}\tilde{q}$, let $z = Z(q, s, \alpha)$ be defined by Lemma 5.2, and let $\mathbf{v} = \mathbf{V}(z, s, \alpha)$ and $\mathbf{e} = \hat{\mathbf{E}}(z, s, \alpha)$ be defined by (5.1) and (5.6), respectively. We then substitute these functions into the right sides of (5.21) and (5.22). The output functions are denoted as $\tilde{q}^\sharp(t)$, $s^\sharp(t)$, and $\alpha^\sharp(t)$. The map $Q \to \Psi(Q) = (\tilde{q}^\sharp, s^\sharp, \alpha^\sharp)$ is the (contraction) mapping.

The following estimates are shown in [11, Lem. 3.1].

$$\|\tilde{q}^\sharp\|_{Str[I]} \lesssim \|q_0\|_{L^2_x} + (\sigma^{\frac{1}{2}} + \|q\|^2_{L^4_{t,x}[I]})\, \|q\|_{L^4_{t,x}[I]} . \quad (5.24)$$

We also have $|\vec{G}_2| \lesssim \|z\|_X + \|z\|_X^4$ and thus

$$|s^\sharp(t) - 1| + |\alpha^\sharp(t) - \alpha_0| \lesssim \int_0^t |\vec{G}_2(\tau)|\, d\tau \lesssim \sigma \|q\|_{L^\infty_t L^2_x} + \sigma \|q\|^4_{L^\infty_t L^2_x} .$$

Therefore $\mathcal{A}_{\delta,\sigma}$ is invariant under the map Ψ if δ and σ are sufficiently small.

We now consider the more delicate difference estimate. Suppose we have $Q_l = (\tilde{q}_l, s_l, \alpha_l)(t)$ for $l = a, b$. Let z_l, v_l, $\mathbf{e}_l$, $\tilde{q}_l^\sharp$, $s_l^\sharp$ and $\alpha_l^\sharp$ be defined respectively. Denote

$$\delta\tilde{q} = \tilde{q}_a(t,r) - \tilde{q}_b(t,r), \quad \delta z = z_a(t, r/s_a) - z_b(t, r/s_b), \quad etc. \quad (5.25)$$

Note that we define δz in terms of r, not in ρ, i.e., $\delta z \ne z_a(\rho) - z_b(\rho)$. See Remark 5.3 after the proof. In the rest of the proof, we denote

$$\|q\|_2 = \max_{a,b}(\|q_a\|_2, \|q_b\|_2), \quad \|z\|_X = \max_{a,b}(\|z_a\|_X, \|z_b\|_X), \quad etc.$$

To start with, note that

$$\|z\|_{L^\infty_t X} \lesssim \delta, \quad |\delta h_1| \lesssim |\delta s|\frac{h_1}{r}, \quad |\delta h_3| \lesssim |\delta s|\frac{h_1^2}{r}, \quad |\delta\gamma| \lesssim |z||\delta z|. \quad (5.26)$$

We first estimate $\delta\mathbf{e} = \mathbf{e}_a - \mathbf{e}_b = \hat{\mathbf{E}}(z_a, s_a, \alpha_a) - \hat{\mathbf{E}}(z_b, s_b, \alpha_b)$. By (5.7),

$$|\delta\mathbf{e}| \lesssim |\delta\alpha| + \|\hat{\mathbf{E}}(z_a(\cdot/s_a)) - \hat{\mathbf{E}}(z_b(\cdot/s_b))\|_{L^\infty} .$$

By Lemma 5.1, $\|\hat{\mathbf{E}}(z_a(\cdot/s_a)) - \hat{\mathbf{E}}(z_b(\cdot/s_b)\|_{L^\infty} \lesssim \|z_a(\cdot/s_a) - z_b(\cdot/s_b)\|_X = \|\delta z\|_X$. Thus

$$|\delta \mathbf{e}| \lesssim |\delta\alpha| + \|\delta z\|_X. \tag{5.27}$$

We next estimate $\|\delta z\|_X$. By (5.3),

$$\delta(L_0 z)\hat{\boldsymbol{\jmath}} = \delta[se^{-\alpha R} q \mathbf{e}(r)] - \delta \mathbf{G_0}.$$

Here $\delta(L_0 z) = L_0(\frac{r}{s_a}) z_a(\frac{r}{s_a}) - L_0(\frac{r}{s_b}) z_b(\frac{r}{s_b})$ and $\delta \mathbf{G_0} = \mathbf{G_0}(z_a(\frac{r}{s_a})) - \mathbf{G_0}(z_b(\frac{r}{s_b}))$. Rewrite

$$\delta(L_0 z) = D_1 + L_0(r/s_a)\delta_1 z,$$

where

$$D_1 = (\delta L_0) z_b(r/s_b), \quad \delta_1 z = z_a(r/s_a) - \Pi_{s_a} z_b(r/s_b)$$

and Π_s is the projection removing $h_1(z/s)$: $\Pi_s f = f - \frac{\langle h_1(\cdot/s), f\rangle_X}{\langle h_1, h_1\rangle_X} h_1(\cdot/s)$. Here we have used $L_0(r/s_a) = L_0(r/s_a)\Pi_{s_a}$. Since $L_0(r/s) = s[\partial_r - \frac{m}{r} h_3(r/s)]$, we have $\delta L_0 \sim \delta s[L_0(r/s) - s\frac{m^2}{r^2} h_1^2(r/s) \cdot \frac{r}{s^2}]$, and hence

$$\|D_1\|_{L^2} \lesssim |\delta s| \cdot \|z\|_X .$$

Thus, taking $L_0(r/s_a)^{-1}$,

$$\|\delta_1 z\|_X \lesssim \left\|\delta[se^{-\alpha R} q \mathbf{e}(r)]\right\|_2 + \|\delta \mathbf{G_0}\|_2 + \|D_1\|_{L^2} .$$

We can decompose

$$\delta z = \delta_1 z + \delta_2 z, \quad \delta_2 z = (1 - \Pi_{s_a}) z_b(r/s_b),$$

and we have

$$\|\delta_2 z\|_X \lesssim \langle h_1(z/s_a) - h_1(z/s_b), z_b(r/s_b)\rangle_X \le |\delta s| \, \|z\|_X .$$

Note

$$\begin{aligned} |\delta \mathbf{G_0}| &\lesssim |\delta \mathbf{h}||\gamma_\rho| + \frac{\delta s}{r}(|\gamma| + |\boldsymbol{\xi}|^2) + |\delta\gamma_\rho| + \frac{1}{r}(|\delta\gamma| + |\boldsymbol{\xi}||\delta\boldsymbol{\xi}|) \\ &\lesssim |\delta s|(|z||z_r| + |z|^2/r) + |\delta z|(|z_r| + |z|/r) + |z||\delta z_r|. \end{aligned}$$

Thus,

$$\|\delta \mathbf{G_0}\|_2 \lesssim |\delta s| \, \|z\|_X^2 + \|z\|_X \, \|\delta z\|_X .$$

Finally,

$$\left\|\delta[se^{-\alpha R} q \mathbf{e}(r)]\right\|_2 \lesssim (|\delta s| + |\delta\alpha| + \|\delta \mathbf{e}\|_{L^\infty}) \cdot \|q\|_2 + \|\delta q\|_2 .$$

Adding these estimates, using (5.27), and $\|z\|_X \lesssim \|q\|_2$, we get

$$\|\delta z\|_X \lesssim (|\delta s| + |\delta\alpha| + \|\delta z\|_X) \cdot \|q\|_2 + \|\delta q\|_2 .$$

Absorbing $\|\delta z\|_X \|q\|_2$ to the left side, we get

$$\|\delta z\|_X \lesssim (|\delta s| + |\delta\alpha|) \cdot \|q\|_2 + \|\delta q\|_2 . \tag{5.28}$$

We now estimate $\left\|\delta\tilde{q}^\sharp\right\|_{Str[I]}$. Apply Strichartz estimate to the difference of (5.21),

$$\begin{aligned}\left\|\delta\tilde{q}^\sharp\right\|_{Str[I]} &\lesssim \|\delta(V\tilde{q})\|_{L^{4/3}_{t,x}} \lesssim \|V(\delta\tilde{q})\|_{L^{4/3}_{t,x}} + \|(\delta V)\tilde{q}\|_{L^{4/3}_{t,x}}. \\ &\lesssim \|V\|_{L^2_{t,x}} \|\delta\tilde{q}\|_{L^4_{t,x}} + \|\delta V\|_{L^2_{t,x}+L^{8/3}_t L^{8/5}_x} \|\tilde{q}\|_{L^4_{t,x}\cap L^{8/3}_t L^8_x}.\end{aligned}$$

Recall $V = V_1 - V_2 + \int^r \frac{1}{r'}V_2$. By the 4-dimensional Hardy inequality, for each fixed t,

$$\|V\|_{L^2_x} \lesssim \|V_1\|_{L^2_x} + \|V_2\|_{L^2_x} \lesssim \|\frac{1+v_3}{r^2}\|_2 + \|\frac{v_{3,r}}{r}\|_2 + \|q\|_4^2 + \|q\|_4 \cdot \|\frac{\nu}{r}\|_4,$$

and, since $v_3(r) = (h_3 + h_3\gamma + h_1 z_2)(r/s)$ and $|\nu| = |\widehat{\mathbf{k}} - v_3\mathbf{v}|$,

$$\|\frac{1+v_3}{r^2}\|_2 + \|\frac{v_{3,r}}{r}\|_2 \lesssim 1 + \|z\|_X + \|\frac{z}{r}\|_4 \cdot \|z\|_{X_4},$$

$$\|\frac{\nu}{r}\|_4^2 = \|\frac{1-v_3^2}{r^2}\|_2 \lesssim \|\frac{1+v_3}{r^2}\|_2.$$

Thus $\|V\|_{L^2_x} \lesssim 1 + \|q\|^2_{L^4_x}$ and hence $\|V\|_{L^2_{t,x}[I]} \lesssim \sigma^{1/2} + \|q\|^2_{L^4_{t,x}}$.

Denote $Y = L^2_{t,x} + L^{8/3}_t L^{8/5}_x$. By Hardy inequality again,

$$\begin{aligned}\|\delta V\|_Y &\lesssim \|\delta V_1\|_Y + \|\delta V_2\|_Y \\ &\lesssim \|\frac{\delta v_3}{r^2}\|_Y + \|\frac{\partial_r \delta v_3}{r}\|_Y + (\|q\|_{L^4_{t,x}} + \|\frac{\nu}{r}\|_{L^4_{t,x}}) \|\delta q\|_{L^4_{t,x}} + \|q\|_{L^{8/3}_t L^8_x} \cdot \|\frac{\delta\nu}{r}\|_{L^\infty_t L^2_x}.\end{aligned}$$

Note $\nu = \mathbf{e}\cdot(\widehat{\mathbf{k}} - v_3\mathbf{v})$. Thus $\delta\nu = \delta\mathbf{e}\cdot(\widehat{\mathbf{k}} - v_3\mathbf{v}) - \mathbf{e}\cdot((\delta v_3)\mathbf{v} + v_3\delta\mathbf{v})$, and

$$\|\frac{\delta\nu}{r}\|_{L^2_x} \lesssim \|\delta\mathbf{e}\|_\infty \|\frac{1}{r}(\widehat{\mathbf{k}} - v_3\mathbf{v})\|_2 + \|\frac{1}{r}\delta\mathbf{v}\|_2.$$

Since $\|\frac{1}{r}(\widehat{\mathbf{k}} - v_3\mathbf{v})\|_2 \lesssim 1 + \|z\|_X^2 \lesssim 1$ and $\|\frac{1}{r}\delta\mathbf{v}\|_2 \lesssim |\delta\alpha| \|\frac{\mathbf{h}+\boldsymbol{\xi}}{r}\|_2 + \|\frac{\delta\mathbf{h}}{r}\|_2 + \|\frac{\delta z}{r}\|_2$, we conclude using (5.27) and (5.28),

$$\|\frac{\delta\nu}{r}\|_{L^2_x} \lesssim |\delta s| + |\delta\alpha| + \|\delta q\|_2.$$

For $\frac{\delta v_3}{r^2}$ and $\frac{\partial_r \delta v_3}{r}$, since $v_3(r) = (h_3 + h_3\gamma + h_1 z_2)(r/s)$,

$$\frac{1}{r^2}|\delta v_3| \lesssim \frac{1}{r^2}(|\delta h_3| + |\delta h_1||z| + |\delta\gamma| + h_1|\delta z|) \lesssim \frac{h_1 + |z|}{r}\left(|\delta s|\frac{h_1}{r} + \frac{|\delta z|}{r}\right),$$

$$\frac{1}{r}|\partial_r \delta v_3| \lesssim |\delta s| \left(\frac{h_1(h_1 + |z|)}{r^2} + \frac{h_1 + h_1^2|z|}{r}|z_r|\right) + \frac{h_1 + h_1^2|z|}{r}\frac{|\delta z|}{r} + \frac{h_1 + |z|}{r}|\partial_r \delta z|.$$

We do not want to bound $\frac{z}{r}\frac{\delta z}{r}$ and $\frac{z}{r}\partial_r \delta z$ in L_x^2 since otherwise we would need a bound for $\|\delta z\|_{X_p}$, $p>2$, which requires extra effort. We have

$$\begin{aligned}\|\frac{\delta v_3}{r^2}\|_Y + \|\frac{\partial_r \delta v_3}{r}\|_Y \lesssim\ & \|\delta s\|_{L_t^\infty} \left\| \left(\frac{h_1(h_1+|z|)}{r^2} + \frac{h_1+h_1^2|z|}{r}|z_r| \right) \right\|_{L^2_{t,x}} \\ & + \left\| \frac{h_1+h_1^2|z|}{r}\frac{|\delta z|}{r} \right\|_{L^2_{t,x}} + \left\| \frac{h_1+|z|}{r}(\frac{|\delta z|}{r} + |\partial_r \delta z|) \right\|_{L_t^{8/3}L_x^{8/5}} \\ \lesssim\ & \|\delta s\|_{L_t^\infty} + (1+\|z/r\|_{L_t^{8/3}L_x^8})\, \|\delta z\|_{L_t^\infty X}\,.\end{aligned}$$

Using $\|z/r\|_{L_t^{8/3}L_x^8} \lesssim \|q\|_{L_t^{8/3}L_x^8} \lesssim \delta$, and (5.28), we conclude

$$\|\delta\tilde q^\sharp\|_{Str[I]} \lesssim (\sigma^{1/2} + \|\tilde q\|^2_{L^4_{t,x}})\, \|\delta \tilde q\|_{L^4_{t,x}} + \|\tilde q\|_{Str[I]}\, (\|\delta s\|_{L_t^\infty} + \|\delta\alpha\|_{L_t^\infty} + \|\delta\tilde q\|_{L_t^\infty L_x^2}). \quad (5.29)$$

We now estimate $\delta s^\sharp$ and $\delta\alpha^\sharp$. Estimating the difference of (5.22),

$$\|\delta s^\sharp\|_{L^\infty(I)} + \|\delta\alpha^\sharp\|_{L^\infty(I)} \lesssim \int_I (|\delta s| + |\delta A|)|\vec G_2| + |\delta \vec G_2| d\tau.$$

Note that $|\vec G_2| \lesssim \|z\|_X + \|z\|_X^4$,

$$|\delta A| \lesssim \|\delta \mathbf{h}\|_X \|z\|_X + \|h_1\|_X \|\delta z\|_X \lesssim |\delta s| \|z\|_X + \|\delta z\|_X,$$

and

$$\begin{aligned}|\delta \vec G_2| &\lesssim \|\delta h\|_X \|z\|_X + \|h_1\|_X\|\delta z\|_X + |\delta G_1| \\ &\lesssim |\delta s|\|z\|_X + \|\delta z\|_X + (1+\|z\|_\infty)\,(\|z\|_\infty \|\partial_r \delta z\|_2 + \|\partial_r z\|_2 \|\delta z\|_\infty) \\ &\quad + (\|z\|_\infty + \|z\|_\infty^3)\|\delta z/r\|_2,\end{aligned}$$

Thus,

$$\begin{aligned}\|\delta s^\sharp\|_{L^\infty(I)} + \|\delta\alpha^\sharp\|_{L^\infty(I)} &\lesssim \int_I |\delta s|\, \|z\|_X + (1+\|z\|_X^3)\, \|\delta z\|_X\, d\tau \\ &\lesssim \sigma\, \|z\|_X\, \|\delta s\|_{L^\infty(I)} + \sigma\, \|\delta z\|_{L_t^\infty X}\,. \end{aligned} \quad (5.30)$$

Combining (5.28), (5.29) and (5.30), we have proved that

$$\|\delta \tilde q^\sharp\|_{Str[I]} + \|\delta s^\sharp\|_{L^\infty(I)} + \|\delta \alpha^\sharp\|_{L^\infty(I)} \lesssim (\sigma^{1/2} + \delta)\left(\|\delta \tilde q\|_{Str[I]} + \|\delta s\|_{L^\infty(I)} + \|\delta\alpha\|_{L^\infty(I)} \right). \quad (5.31)$$

Thus Ψ is a contraction mapping on $\mathcal{A}_{\delta,\sigma}$ if σ and δ are sufficiently small. We have therefore established the unique existence of a triplet $[s_W(t), \alpha_W(t), q_W(t)]$ solving the (s,α,q)-system. This yields a map $\mathbf{u}_W(t) \in C([0,T]; \Sigma_m)$.

If $\mathbf{u}_0 \in \dot H^2$, the a priori estimates in [11, Lem. 3.1] show $\|\nabla \tilde q\|_{Str[I]}$ is uniformly bounded, so $\mathbf{u}_W(t) \in C(I; \Sigma_m \cap \dot H^2)$.

If $\mathbf{u}_0^n \to \mathbf{u}_0$ in $\Sigma_m \cap \dot{H}^k$, $k=1,2$, a difference estimate similar to (5.31) shows

$$D^n \lesssim \|\tilde{q}_0^n - \tilde{q}_0\|_2 + (\sigma^{1/2}+\delta)D^n.$$

where $D^n = \|\tilde{q}^n - \tilde{q}\|_{Str[I]} + \|s^n - s\|_{L^\infty(I)} + \|\alpha^n - \alpha\|_{L^\infty(I)}$. Thus $D_n \to 0$ as $n \to \infty$, and hence $\mathbf{u}_W^n \to \mathbf{u}_W$.

The energy $\mathcal{E}(\mathbf{u}_W(t))$ is conserved since $\mathcal{E}(\mathbf{u}_W(t)) = 4\pi m + \pi \|q(t)\|_{L_x^2}^2 = 4\pi m + \pi \|q_0\|_{L_x^2}^2$.

Finally, we must verify that $\mathbf{u}_W$ is a solution of the Schrödinger flow as in Definition 1.2. To do this, approximate the initial data $\mathbf{u}_0$ in Σ_m by $\mathbf{u}_0^k$ with $\nabla \mathbf{u}_0^k \in H^{10}$ (say). By [23] there is a unique strong solution $\mathbf{u}_S^k(t)$ with initial data $\mathbf{u}_0^k$. The corresponding triple $[s_S^k(t), \alpha_S^k(t), q_S^k(t)]$ must satisfy the (s,α,q)-system. By uniqueness, $s_S^k(t) \equiv s_W^k(t)$, etc., and so $\mathbf{u}_W^k(t) \equiv \mathbf{u}_S^k(t)$. By continuous dependence on $\dot{H}^1$ data, $\mathbf{u}_S^k$ converges to $\mathbf{u}_W$ in $C([0,T];\Sigma_m)$, and in particular in $C([0,T];L^2_{loc})$. Finally, $\mathbf{u}_S^k$ satisfies the weak form of the Schrödinger flow (Definition 1.2), and passing to the limit, so does $\mathbf{u}_W$. Dropping the subscript W ($\mathbf{u} := \mathbf{u}_W$), Theorem 1.4 is established.

We now consider Corollary 1.5. Suppose T is the blow-up time. By Theorem 1.4, for each $t < T$ we have $T - t \ge \sigma s(\mathbf{u}(t))^2$. Thus $s(\mathbf{u}(t)) \le \sigma^{-1/2}\sqrt{T-t}$. If $k=2$, by [11, Th. 2.1], $\|\mathbf{u}(t)\|_{\dot{H}^2} \ge C_2/s(\mathbf{u}(t)) \ge C_2\sigma^{1/2}(T-t)^{-1/2}$. On the other hand, the $\dot{H}^2$-estimates of [11] show that the $\dot{H}^2$-norm can only blow-up if $\liminf_{t\to T^-} s(t) = 0$. Thus $T^2_{max} = T^1_{max}$. Statement (ii) follows from Theorem 1.4 directly. Corollary 1.5 is established.

Remark 5.3 1. In Theorem 1.4, we did not try to prove continuity on data $\mathbf{u}_0$ in $\dot{H}^2$, which would require difference estimates in H^1 for $\tilde{q}$.

2. In (5.25), we define δz in terms of r, not in ρ, i.e., $\delta z \ne \tilde{\delta} z = z_a(\rho) - z_b(\rho)$. Indeed, in view of (5.3), since L_0 depends on ρ, it may seem natural to bound $\tilde{\delta} z$ using $L_0 \tilde{\delta} z \hat{\boldsymbol{\jmath}} = \delta[s e^{-\alpha R} q \mathbf{e}(s\rho)] + \tilde{\delta}\mathbf{G_0}$. However, to bound the right side we need to bound the difference $q_a \mathbf{e}_a(s_b\rho) - q_a\mathbf{e}_a(s_a\rho) = \int_{s_a}^{s_b} \rho \partial_r(q_a\mathbf{e}_a)(\sigma\rho)d\sigma$, for which $\|\mathbf{u}\|_{\dot{H}^2}$ is insufficient and we need a weighted norm of $\mathbf{u}$. The reason is that the dilation magnifies the difference when ρ is large. In addition, to bound δv_3 using $\tilde{\delta} z$ instead of δz, one needs a bound for z_{rr}.

3. In the proof we have avoided using $\|\delta z\|_{X_4}$ since its estimate requires $\|\delta \mathbf{e}\|_\infty$. We know how to control $\|\delta\mathbf{e}\|_\infty$ by $\|\delta z\|_X$, but we do not know if $\|\delta \mathbf{e}\|_\infty \lesssim \|\delta z\|_{X_4}$.

5.4 Small energy case

The proof of Theorem 5.2 is similar to that of Theorem 1.4.

Proof. When $m \ge 1$, the limits $\lim_{r\to 0}\mathbf{v}_0(r)$ and $\lim_{r\to\infty}\mathbf{v}_0(r)$ exist and it is necessary that $\mathbf{u}_0(0) = \mathbf{u}_0(\infty)$. We may assume $\mathbf{u}_0(0) = \mathbf{u}_0(\infty) = -\hat{\mathbf{k}}$. In the proof for Theorem 1.4, we may redefine

$$\mathbf{h}(r) := -\hat{\mathbf{k}}, \quad \mathbf{v}(r) = (z_2, z_1, -1-\gamma)^T,$$

and the parameters s and α are no longer needed. The same proof, in particular the difference estimate $\|\delta \tilde{q}^\sharp\|_{Str[I]} \lesssim (\sigma^{1/2}+\delta)\|\delta\tilde{q}\|_{Str[I]}$, then gives the local wellposedness. □

Note that this proof does not directly apply to the radial case, since $\|\mathbf{u}(r)\|_{\dot{H}^1}$ no longer controls $\|z/r\|_2$.

6 Appendix: some lemmas

6.1 Computation of nonlinear terms

To find the equations for $\dot{s}$ and $\dot{\alpha}$, we need to compute $(g, (\mathbf{V^h})^{-1} P^{\mathbf{h}} \mathbf{F_1})_{L^2}$ for $g = h_1$ or $g = N_0 h_1$. Here is the result.

Lemma 6.1 *Recall* $\mathbf{F_1} = -2\gamma_r \frac{m}{r} h_1 \widehat{\boldsymbol{\jmath}} + \xi \times (\Delta_r + \frac{m^2}{r^2} R^2 \xi)$ *and* $(\mathbf{V^h})^{-1} P^{\mathbf{h}} \mathbf{F_1} = \widehat{\boldsymbol{\jmath}} \cdot \mathbf{F_1} + i(h \times \widehat{\boldsymbol{\jmath}}) \cdot \mathbf{F_1}$. *For any suitable function g,*

$$(g, (\mathbf{V^h})^{-1} P^{\mathbf{h}} \mathbf{F_1})_{L^2} = \int_0^\infty \bigg(i g_r(-\gamma z_r + z\gamma_r) + \frac{m}{r} h_1 g(-2\gamma_r - i z_2 z_{1,r} + i z_1 z_{2,r}) \\ + \frac{m}{r}(h_1 g)_r(\gamma^2 - i z_2 z) + i\frac{m^2}{r^2}(2h_1^2 - 1) g \gamma z \bigg) r dr. \tag{6.1}$$

Proof. Decompose

$$\int_0^\infty g (\mathbf{V^h})^{-1} P^{\mathbf{h}} \mathbf{F_1} r dr = \int -2g\frac{m}{r} h_1 \gamma_r + \int g P(\xi \times \Delta_r \xi) + \int g P(\xi \times \frac{m^2}{r^2} R^2 \xi) =: I_1 + I_2 + I_3.$$

Denote $[a, b, c] = a\widehat{\boldsymbol{\jmath}} + bh \times \widehat{\boldsymbol{\jmath}} + ch$. For any vector η,

$$P(\xi \times \eta) = [1, i, 0] \cdot ([z_1, z_2, \gamma] \times \eta) = ([1, i, 0] \times [z_1, z_2, \gamma]) \cdot \eta = [i\gamma, -\gamma, -iz] \cdot \eta.$$

Since $h_r = \frac{m}{r} h_1 h \times \widehat{\boldsymbol{\jmath}}$,

$$\partial_r [a, b, c] = [a_r,\ b_r + \frac{m}{r} h_1 c,\ c_r - \frac{m}{r} h_1 b].$$

Thus

$$\begin{aligned} I_2 &= \int g[i\gamma, -\gamma, -iz] \cdot \Delta_r [z_1, z_2, \gamma] \\ &= \int \partial_r [-ig\gamma, g\gamma, igz] \cdot \partial_r [z_1, z_2, \gamma] \\ &= \int [-i(g\gamma)_r, (g\gamma)_r + \frac{m}{r} h_1 igz,\ i(gz)_r - \frac{m}{r} h_1 g\gamma] \cdot [z_{1,r},\ z_{2,r} + \frac{m}{r} h_1 \gamma,\ \gamma_r - \frac{m}{r} h_1 z_2] \\ &= \int g(-i\gamma_r z_{1,r} + \gamma_r z_{2,r} + i z_r \gamma_r) + \int g_r(-i\gamma z_{1,r} + \gamma z_{2,r} + iz\gamma_r) \\ &\quad + \int g_r(\frac{m}{r} h_1 \gamma^2 - i\frac{m}{r} h_1 z_2 z) + \int g(-i\frac{m}{r} h_1 z_2 z_r + i\frac{m}{r} h_1 z z_{2,r}) + \int g\frac{m^2}{r^2} h_1^2 i z_1 \gamma. \end{aligned}$$

Note that the first integral is zero, and we have canceled two $\int g\frac{m}{r}h_1\gamma\gamma_r$. Also,

$$
\begin{aligned}
I_3 &= \int g[i\gamma, -\gamma, -iz] \cdot \frac{m^2}{r^2}R^2\xi = \int g(\gamma h_3 - ih_1 z,\ i\gamma,\ *) \cdot \frac{m^2}{r^2}(z_2 h_3 - \gamma h_1,\ z_1,\ 0) \\
&= \int \frac{m^2}{r^2} g(h_3^2\gamma z_2 - h_1 h_3 \gamma^2 + ih_1 h_3 z_2 z + ih_1^2\gamma z - i\gamma z_1).
\end{aligned}
$$

Summing up $I_1 + I_2 + I_3$, we get the Lemma. $\square$

6.2 Linear weighted-L^2 estimate

Lemma 6.2 *Let H be a self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying the weighted resolvent estimate*

$$
\sup_{\mu\notin\mathbb{R};\ \phi\in L^2, \|\phi\|_{L^2}=1} \left\| \frac{1}{|x|}(H-\mu)^{-1}\frac{1}{|x|}\phi \right\|_{L^2} \lesssim 1.
$$

Then for $f(x,t) \in \frac{1}{|x|}L^2$,

$$
\left\| \frac{1}{|x|}\int_0^t e^{i(t-s)H} f(x,s)ds \right\|_{L^2_{x,t}(\mathbb{R}^n\times\mathbb{R})} \lesssim \| |x| f\|_{L^2_{x,t}(\mathbb{R}^n\times\mathbb{R})}.
$$

Proof. First some simplifications. It suffices to prove the estimate for $f(x,t)$ compactly supported, and $f(x,t) \in \frac{1}{|x|}L^2_{x,t} \cap L^\infty_t L^2_x$ (by density). Also, it is enough to consider $t \geq 0$ (i.e. $f(x,t)$ supported in $\{t \geq 0\}$). Finally, we regularize the integral: set

$$
F_\epsilon(x,t) := \frac{1}{|x|}\int_0^t e^{i(t-s)(H+i\epsilon)} f(x,s)ds.
$$

We will prove the estimate for F_ϵ with an ϵ-independent constant, and the lemma follows from this. Under our assumptions, F_ϵ is well-defined as a $\frac{1}{|x|}L^2_x$-valued function of t, and

$$
\int_0^\infty \| |x| F_\epsilon(\cdot, t)\|_{L^2_x} dt < \infty.
$$

Hence the Fourier transform of F_ϵ in t is well-defined (as a $\frac{1}{|x|}L^2_x$ -valued function of τ):

$$
\tilde{F}_\epsilon(x,\tau) := (2\pi)^{-1/2}\int_0^\infty e^{-it\tau}F_\epsilon(x,t)dt.
$$

Changing the order of integration, we see

$$
\begin{aligned}
\tilde{F}_\epsilon(x,\tau) &= \frac{1}{|x|}(2\pi)^{-1/2}\int_0^\infty dt e^{-it\tau}\int_0^t ds e^{i(t-s)(H+i\epsilon)} f(x,s) \\
&= \frac{1}{|x|}(2\pi)^{-1/2}\int_0^\infty ds e^{-is(H+i\epsilon)}\int_s^\infty dt e^{it(H-\tau+i\epsilon)} f(x,s) \\
&= \frac{1}{|x|}(2\pi)^{-1/2}(i)(H-\tau+i\epsilon)^{-1}\int_0^\infty ds e^{-is\tau} f(x,s)ds \\
&= \frac{1}{|x|}(i)(H-\tau+i\epsilon)^{-1}\tilde{f}(x,\tau)
\end{aligned}
$$

and so using the weighted resolvent estimate gives

$$\|\tilde{F}_\epsilon\|_{L^2_x} \lesssim \||x|\tilde{f}(x,\tau)\|_{L^2_x},$$

and squaring and integrating in τ yields

$$\|\tilde{F}_\epsilon\|^2_{L^2_{x,\tau}} \lesssim \||x|\tilde{f}\|^2_{L^2_{x,\tau}} \lesssim \||x|f\|^2_{L^2_{x,t}}.$$

By a vector-valued version of the Plancherel theorem (see eg. [20], Ch. XIII.7),

$$\|F_\epsilon\|^2_{L^2_{x,t}} = \|\tilde{F}_\epsilon\|^2_{L^2_{x,t}} \lesssim \||x|f\|^2_{L^2_{x,t}},$$

completing the proof. □

6.3 Proof of the splitting lemma

Here we prove Lemma 4.1.

Proof. For $\mathbf{u} = e^{m\theta R}\mathbf{v}(r) \in \Sigma_m$, $s > 0$, and $\alpha \in \mathbb{R}$, define

$$F(\mathbf{u}; s, \alpha) := \int_0^\infty (\hat{\boldsymbol{\jmath}} + iJ^{\mathbf{h}(\rho)}\hat{\boldsymbol{\jmath}}) \cdot e^{-\alpha R}\mathbf{v}(s\rho)h_1(\rho)\rho d\rho \quad \in \mathbb{C}.$$

Note that for $\mathbf{u}$ of the form (4.1), (4.2) is equivalent to $F(\mathbf{u}; s, \alpha) = 0$.

Suppose $\mathcal{E}(\mathbf{u}) \le 4\pi m + \delta^2$. It is shown in [11] that if δ is sufficiently small, then there are $\hat{s}$, $\hat{\alpha}$, and $\hat{z}$ such that $\mathbf{u}(r,\theta) = e^{[m\theta+\hat{\alpha}]R}[(1+\hat{\gamma}(r/\hat{s}))\mathbf{h}(r/\hat{s}) + \mathbf{V}^{r/\hat{s}}(\hat{z}(r/\hat{s}))]$, and with $\|\hat{z}\|^2_X \lesssim \delta_1^2 := \mathcal{E}(\mathbf{u}(0)) - 4\pi m \le \delta^2$ (but $\hat{z}$ does not satisfy (4.2)).

It follows from this, and the fact that $\rho h_1(\rho) \in L^2(\rho d\rho)$ for $m \ge 3$, that for some $\delta_0 > 0$, F is a C^1 map from

$$\{\mathbf{u} \in \Sigma_m \mid \mathcal{E}(\mathbf{u}) \le 4\pi m + \delta_0^2\} \times (\mathbb{R}^+ \times \mathbb{R})$$

into $\mathbb{C}$. Furthermore, straightforward computations show that

$$F(e^{m\theta R}\mathbf{h}(r); 1, 0) = 0,$$

and

$$\begin{pmatrix} \partial_s F(e^{m\theta R}\mathbf{h}(r); 1, 0) \\ \partial_\alpha F(e^{m\theta R}\mathbf{h}(r); 1, 0) \end{pmatrix} = \|h_1\|^2_{L^2} \begin{pmatrix} i \\ -1 \end{pmatrix}.$$

By the implicit function theorem, we can solve $F = 0$ to get $s = s(\mathbf{u})$ and $\alpha = \alpha(\mathbf{u})$ for $\mathbf{u}$ in a $\dot{H}^1$-neighbourhood of the harmonic map $e^{m\theta R}\mathbf{h}(r)$.

Since $\|\hat{z}\|_X \lesssim \delta$, provided δ is chosen small enough (depending on the size of this neighbourhood),

$$\hat{\mathbf{u}}(x) := e^{-\hat{\alpha}R}\mathbf{u}(\hat{s}x) = e^{m\theta R}[(1+\hat{\gamma}(r))\mathbf{h}(r) + \mathbf{V}^r\hat{z}(r)]$$

lies in this neighbourhood, yielding $s(\hat{\mathbf{u}})$ and $\alpha(\hat{\mathbf{u}})$ with $F(\hat{\mathbf{u}}; s(\hat{\mathbf{u}}), \alpha(\hat{\mathbf{u}})) = 0$. Furthermore,

$$|s(\hat{\mathbf{u}}) - 1| + |\alpha(\hat{\mathbf{u}})| \lesssim \|\hat{z}\|_X,$$

and so

$$\|z(\rho)\|_X = \|(\widehat{\jmath} + iJ^{\mathbf{h}(\rho)}\widehat{\jmath}) \cdot e^{-\alpha(\hat{\mathbf{u}})R}\hat{\mathbf{v}}(s(\hat{\mathbf{u}})\rho)\|_X \lesssim \|\hat{z}\|_X \lesssim \mathcal{E}(\mathbf{u}) - 4\pi m.$$

To complete the proof of the lemma, undo the rescaling: set $s(\mathbf{u}) := s(\hat{\mathbf{u}})/\hat{s}$ and $\alpha(\mathbf{u}) := \alpha(\hat{\mathbf{u}}) + \hat{\alpha}$. $\square$

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